

# Hyperplane Configurations, Subtractive Sets, their Groupoids, and their Blowups

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## Definition

A finite set  $Z$  in  $\mathbb{P}_k^n$  is **geproci** if the projection  $\bar{Z}$  of  $Z$  from a general point  $P$  to a hyperplane  $H = \mathbb{P}_k^{n-1}$  is a complete intersection in  $H$ .

Geproci stands for **g**eneral **p**rojection is a **c**omplete **i**ntersection. The only nontrivial examples known are for  $n = 3$ . In this case a hyperplane is a plane. A reduced set of points in a plane is a complete intersection if it is the transverse intersection of two algebraic curves. For  $\#Z = ab$  ( $a \leq b$ ),  $Z$  is  $(a, b)$ -geproci if  $\bar{Z}$  is the intersection of a degree  $a$  curve and a degree  $b$  curve.

## Definition

Given a  $k \geq 0$  and field  $F$ , a  **$k$ -spread** of  $\mathbb{P}_F^n$  is a set of mutually-skew  $k$ -planes that partition  $\mathbb{P}_F^n$ .

Spreads are known to exist for  $F =$  any finite field if and only if  $k + 1$  divides  $n + 1$ , and also for  $k + 1 = (n + 1)/2$  for  $F = \mathbb{R}$ .

Spreads are instrumental for the proof that  $\mathbb{P}_{\mathbb{F}_q}^3$  is geproci in  $\mathbb{P}_{\mathbb{F}_q}^3$  [K– 24].

In this case, a spread is a partition of  $\mathbb{P}_{\mathbb{F}_q}^3$  into lines.

# Gorla's Construction

It is possible to construct a spread of  $k$ -dimensional hyperplanes inside  $\mathbb{P}_{\mathbb{F}_q}^n$  if and only if  $k + 1$  divides  $n + 1$ .

Let  $p(x) \in \mathbb{F}_q[x]$  be irreducible, monic, and degree  $k + 1$ , and let  $P$  be its companion matrix. Then one can construct a spread of spaces of the form

$$\text{row} \left( \underbrace{0 \quad \cdots \quad 0 \quad \boxed{I_{k+1}} \quad A_1 \quad \cdots \quad A_j}_{\frac{n+1}{k+1}} \right) : A_i \in \mathbb{F}_q[P].$$

# Maximal Partial Spreads

Note that a spread over  $\mathbb{P}_{\mathbb{F}_q}^3$  comprises  $q^2 + 1$  mutually-skew lines.

## Definition

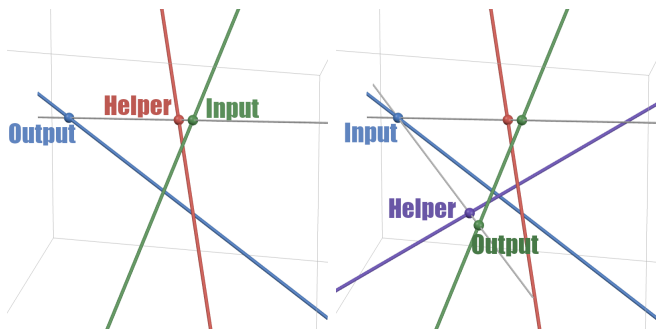
A **partial spread** of  $\mathbb{P}_{\mathbb{F}_q}^3$  with **deficiency**  $d$  is a set of  $q^2 + 1 - d$  mutually-skew lines. A **maximal partial spread** is a partial spread of positive deficiency that is not contained in any larger partial spread.

Maximal partial spreads are also instrumental to the study of geproci sets; in fact, given a maximal partial spread  $\mathcal{M}$ , the set  $\mathbb{P}_{\mathbb{F}_q}^3 \setminus (\bigcup_{L \in \mathcal{M}} L)$  is geproci [K– 24].

# Projecting a Line to a Line via a Line

## Definition

Given three lines  $L_1, L_2, L_3 \subseteq \mathbb{P}_k^3$  where  $L_1 \cap L_2 = L_2 \cap L_3 = \emptyset$ , we can define the function  $\pi(L_1, L_2, L_3) : L_1 \rightarrow L_3$  as follows: take  $p \in L_1$ . Then there is a unique line  $T$  such that  $p \in T$ ,  $T \cap L_2 \neq \emptyset$ , and  $T \cap L_3 \neq \emptyset$ . Then define  $\pi(L_1, L_2, L_3)(p) = T \cap L_3$ . This is the **projection** of  $L_1$  to  $L_3$  **via**  $L_2$ .



## Definition

A **groupoid** is a category  $\mathcal{G}$  where every morphism is invertible.

- For any object  $G \in \mathcal{G}$ ,  $\text{Hom}_{\mathcal{G}}(G, G) = \text{Aut}_{\mathcal{G}}(G)$  is a group.

$\text{Aut}_{\mathcal{G}}(G)$  is a “group of the groupoid.”

- Whenever  $\text{Hom}_{\mathcal{G}}(G_1, G_2) \neq \emptyset$ , then  $\text{Aut}_{\mathcal{G}}(G_1) \cong \text{Aut}_{\mathcal{G}}(G_2)$ .

So when  $\text{Hom}_{\mathcal{G}}(G_1, G_2) \neq \emptyset$  for all  $G_1, G_2 \in \mathcal{G}$ ,  $\mathcal{G}$  induces only one group of the groupoid, up to isomorphism. Then it makes sense to say “the” group of the groupoid,  $\text{Aut}_{\mathcal{G}}$ .

## Theorem

Let  $\mathcal{L}$  be a set of lines in  $\mathbb{P}_F^3$ . Define  $\Pi$  to be the composition-closure of the set of functions  $\{\pi(L_i, L_j, L_k) : L_i, L_j, L_k \in \mathcal{L}, L_i \cap L_j = L_j \cap L_k = \emptyset\}$ . Then  $(\mathcal{L}, \Pi)$  is a groupoid.

In this case, any group of the groupoid is a subgroup of  $\text{Aut}(\mathbb{P}_F^1) \cong \text{PGL}(2, F)$ .

What can we say about this groupoid and its corresponding group(s)? In characteristic 0, when is it finite versus infinite? When does  $\text{Aut}_{(\mathcal{L}, \Pi)}(L)$  have finite orbits, or finitely many orbits? What is the relationship (if  $\exists$ ) between  $\text{Aut}_{(\mathcal{L}, \Pi)} \leq \text{Aut}(\mathbb{P}^1)$  and  $\text{Aut}(\mathcal{L}) \leq \text{Aut}(\mathbb{P}^3)$ ? In characteristic 0, it is known that when  $\text{Aut}_{(\mathcal{L}, \Pi)}$  is finite, then orbits of points under this groupoid are geproci [CFFHMSS 25+].

# Ganger's Results

In her 2024 thesis, Ganger used the technique of **transversals** to prove the following theorem:

## Theorem (Ganger Corollary 2.5)

*The group of the groupoid for the Hopf spread induced by the degree-2 field extension  $\mathbb{F}_{q^2}/\mathbb{F}_q$  over a finite field is isomorphic to the quotient  $\mathbb{F}_{q^2}^*/\mathbb{F}_q^* \cong C_{q+1}$ .*

## Definition

Given a set of lines  $\mathcal{L}$  in  $\mathbb{P}_F^3$ , a **transversal** is a line  $T$  in  $\mathbb{P}_F^3$  such that  $T \cap \bar{L} \neq \emptyset$  for all  $L \in \mathcal{L}$ .

The Hopf spread has exactly two transversals  $T_1, T_2$  for any finite field. The intersection of transversal with a line  $L \in \mathcal{S}$  is a fixed point of  $\text{Aut}_{(\mathcal{S}, \Pi)}(L)$ !

# Representing $\text{Aut}_{\mathcal{G}} \leq \text{PGL}(2, F)$

Let  $U = \overline{u_0 u_1}$ ,  $V = \overline{v_0 v_1}$ ,  $W = \overline{w_0 w_1}$  be lines in  $\mathbb{P}^3$ . Then there is a unique matrix  $A \in \text{PGL}(2)$  such that the following diagram commutes.

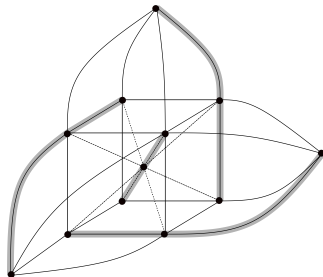
$$\begin{array}{ccc} \mathbb{P}^1 & \xrightarrow{A} & \mathbb{P}^1 \\ \downarrow \overline{u_0 u_1} & & \downarrow \overline{w_0 w_1} \\ U & \xrightarrow{\pi(U, V, W)} & W \end{array}$$

$A$  can be taken to be  $\begin{pmatrix} -u_0 \wedge v_0 \wedge v_1 \wedge w_1 & -u_1 \wedge v_0 \wedge v_1 \wedge w_1 \\ u_0 \wedge v_0 \wedge v_1 \wedge w_0 & u_1 \wedge v_0 \wedge v_1 \wedge w_0 \end{pmatrix}$ .

## Example: The $D_4$ configuration

### Theorem (K-)

Let  $\mathcal{L}$  be the 16 lines of the  $(12_4, 16_3)$  configuration  $D_4$  and let  $\Pi$  be the composition-closure of the projection functions. Then the group of the groupoid  $\mathcal{G} = (\mathcal{L}, \Pi)$  is  $\text{Aut}_{\mathcal{G}} \cong S_3$ .



In fact, one can find a subset of six lines whose group of the groupoid is  $S_3$  [K- 25+]! Such a group is impossible to find in skew configurations [Favacchio 25+].

# Other Finite Groupoids

Using a similar analysis, we have found finite groupoids with automorphism groups isomorphic to  $A_4$  and  $S_4$  within the Penrose and Klein configurations [K– 25+]. These are nontrivial (5, 8)- and (6, 10)-geproci configurations, respectively.

(1, 0, 0, 0)	(0, 1, 0, 0)	(0, 0, 1, 0)	(0, 0, 0, 1)
(0, 1, -1, 1)	(1, 0, -1, -1)	(1, -1, 0, 1)	(1, 1, 1, 0)
(0, 1, -t, t <sup>2</sup> )	(1, 0, -t, -t <sup>2</sup> )	(1, -t, 0, t <sup>2</sup> )	(1, t, t <sup>2</sup> , 0)
(0, 1, -t <sup>2</sup> , t)	(1, 0, -t <sup>2</sup> , -t)	(1, -t <sup>2</sup> , 0, t)	(1, t <sup>2</sup> , t, 0)
(0, 1, -t, 1)	(1, 0, -1, -t)	(1, -t <sup>2</sup> , 0, t <sup>2</sup> )	(1, t, 1, 0)
(0, 1, -t <sup>2</sup> , t <sup>2</sup> )	(1, 0, -t, -1)	(1, -1, 0, t)	(1, t <sup>2</sup> , t <sup>2</sup> , 0)
(0, 1, -1, t)	(1, 0, -t <sup>2</sup> , -t <sup>2</sup> )	(1, -t, 0, 1)	(1, 1, t, 0)
(0, 1, -t <sup>2</sup> , 1)	(1, 0, -1, -t <sup>2</sup> )	(1, -t, 0, t)	(1, t <sup>2</sup> , 1, 0)
(0, 1, -1, t <sup>2</sup> )	(1, 0, -t, -t)	(1, -t <sup>2</sup> , 0, 1)	(1, 1, t <sup>2</sup> , 0)
(0, 1, -t, t)	(1, 0, -t <sup>2</sup> , -1)	(1, -1, 0, t <sup>2</sup> )	(1, t, t, 0)

(0, 0, 1, 1)	(0, 0, 1, i)	(0, 0, 1, -1)	(0, 0, 1, -i)
(0, 1, 0, 1)	(0, 1, 0, i)	(0, 1, 0, -1)	(0, 1, 0, -i)
(0, 1, 1, 0)	(0, 1, i, 0)	(0, 1, -1, 0)	(0, 1, -i, 0)
(1, 0, 0, 1)	(1, 0, 0, i)	(1, 0, 0, -1)	(1, 0, 0, -i)
(1, 0, 1, 0)	(1, 0, i, 0)	(1, 0, -1, 0)	(1, 0, -i, 0)
(1, 1, 0, 0)	(1, i, 0, 0)	(1, -1, 0, 0)	(1, -i, 0, 0)
(1, 0, 0, 0)	(0, 1, 0, 0)	(0, 0, 1, 0)	(0, 0, 0, 1)
(1, 1, 1, 1)	(1, 1, 1, -1)	(1, 1, -1, 1)	(1, 1, -1, -1)
(1, -1, 1, 1)	(1, -1, 1, -1)	(1, -1, -1, 1)	(1, -1, -1, -1)
(1, 1, i, i)	(1, 1, i, -i)	(1, 1, -i, i)	(1, 1, -i, -i)
(1, -1, i, i)	(1, -1, i, -i)	(1, -1, -i, i)	(1, -1, -i, -i)
(1, i, 1, i)	(1, i, 1, -i)	(1, -i, 1, i)	(1, -i, 1, -i)
(1, i, -1, i)	(1, i, -1, -i)	(1, -i, -1, i)	(1, -i, -1, -i)
(1, i, i, 1)	(1, i, -i, 1)	(1, -i, i, 1)	(1, -i, -i, 1)
(1, i, i, -1)	(1, i, -i, -1)	(1, -i, i, -1)	(1, -i, -i, -1)

# On to Higher Dimensions!

Giuseppe Favacchio and I are now interested in studying the groupoid of mutually-skew configurations of  $n - 1$ -planes in  $\mathbb{P}^{2n-1}(F)$  of the form

$$\{H(M) := \text{row}(I_n \ M), H_\infty := \text{row}(0 \ I_n) : M \in F^{n \times n}\}.$$

Then  $\pi_{ijk} : H(M_i) \xrightarrow{H(M_j)} H(M_k)$  is represented by the matrix

$$(M_k - M_j)^{-1}(M_i - M_j) \text{ [Favacchio 25+]}.$$

We are now working on applying this construction in higher dimensions. Note  $H(M) \cap H(M') = \emptyset$  iff  $\det(M - M') \neq 0$ . A set of matrices meeting this criterion is called **subtractive**.

# Blowing Up the Lines

Consider the lines  $U, V, W \subseteq \mathbb{P}^3$ , the blowup map  $\beta : \text{Bl}_W(\mathbb{P}^3) \rightarrow \mathbb{P}^3$ , and the exceptional locus  $E = \beta^{-1}(W) \cong \mathbb{P}^1 \times \mathbb{P}^1$ . Then the projection  $\pi(U, V, W) : U \rightarrow W$  extends to a map  $\tilde{\pi} : U \rightarrow E$ ; furthermore, the image  $\tilde{\pi}(U)$  is a bidegree-(1, 1) curve in  $E$ .

## Example

Consider  $\mathcal{L} = \left\{ H_\infty, H(0), H(I_2), H \begin{pmatrix} \zeta_6 & 1 \\ 0 & \zeta_6^5 \end{pmatrix}, H \begin{pmatrix} \zeta_6 & 0 \\ 1 & \zeta_6^5 \end{pmatrix} \right\}$  in  $\mathbb{P}^3_{\mathbb{C}}$ . Then  $\text{Aut}_{\mathcal{L}} \cong A_4$  [Favacchio 25+], and the six induced bidegree-(1,1) curves in  $E_\infty$  have a stabilizer group  $S_4 \rtimes C_2$ .