Groupoids of Configurations of Lines

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Background: Geproci

Definition

A finite set Z in $\mathbb{P}^n_k$ is **geproci** if the projection $\overline{Z}$ of Z from a general point P to a hyperplane $H = \mathbb{P}^{n-1}_k$ is a complete intersection in H.

Geproci stands for **ge**neral **pro**jection is a **c**omplete **i**ntersection. The only nontrivial examples known are for $n = 3$. In this case a hyperplane is a plane. A reduced set of points in a plane is a complete intersection if it is the transverse intersection of two algebraic curves. For $\#Z = ab$ $(a \leq b)$, Z is (a, b)-geproci if $\overline{Z}$ is the intersection of a degree a curve and a degree b curve.

Background: k-Spreads in $\mathbb{P}^n_F$

Definition

Given a $t \geq 0$ and field F, a k-**spread** of $\mathbb{P}^n_F$ is a set of mutually-skew k-planes that partition $\mathbb{P}^n_F$.

Spreads are known to exist for $F =$ any finite field if and only if $k+1$ divides $n+1$, also and for $k+1 = (n+1)/2$ for $F = \mathbb{R}$.
Spreads are instrumental for the proof that $\mathbb{P}^3_{\mathbb{F}_q}$ is geproci under $\mathbb{P}^3_{\overline{\mathbb{F}}_q}$. In this case, a spread is a partition of $\mathbb{P}^3_{\mathbb{F}_q}$ into lines.

The Hopf Fibration over $\mathbb{R}$

The Hopf fibration $H : S^3 \to S^2$ can yield a spread over $\mathbb{P}^3_{\mathbb{R}}$.

$$\begin{array}{ccc} S^3 & \xrightarrow{H} & S^2 \\ \downarrow{\scriptstyle A} & & \downarrow{\scriptstyle \cong} \\ \mathbb{P}^3_{\mathbb{R}} & \xrightarrow{F} & \mathbb{P}^1_{\mathbb{C}} \end{array}$$

Let $L_{a,b}$ denote the line joining the points $(1, 0, a, b)$ and $(0, 1, -b, a)$, and let L_∞ denote the line joining $(0, 0, 1, 0)$ and $(0, 0, 0, 1)$. Then $\mathcal{S} = \{L_{a,b} : a, b \in \mathbb{R}\} \cup \{L_\infty\}$ the the spread yielded by the Hopf fibration.

Note that $L_{a,b}$ and $L_{c,d}$ are indeed skew for $(a,b) \neq (c,d)$. We can see this because

$$\begin{vmatrix} 1 & 0 & a & b \\ 0 & 1 & -b & a \\ 1 & 0 & c & d \\ 0 & 1 & -d & c \end{vmatrix} = (a-c)^2 + (b-d)^2,$$

which can only be 0 if $(a,b) = (c,d) \in \mathbb{R}^2$.

Furthermore, the point $(a,b,c,d) \in \mathbb{P}^3_{\mathbb{R}}$ is on the line $L_{\frac{ac+bd}{a^2+b^2},\frac{ad-bc}{a^2+b^2}}$ if $(a,b) \neq (0,0)$, and on L_∞ otherwise.
So this is indeed a spread over $\mathbb{P}^3_{\mathbb{R}}$!

Spreads over $\mathbb{F}_q$

Since $\mathbb{F}_q$ is not algebraically closed, we can mimic the construction of the Hopf spread!

- First let q be odd. Then there is some $\theta \in \mathbb{F}_q$ such that $x^2 - \theta \in \mathbb{F}_q[x]$ is irreducible. Defining $L_{a,b} = \overline{(1,0,a,b),(0,1,\theta b,a)}$ for $(a,b) \in \mathbb{F}_q$ and $L_\infty = \overline{(0,0,1,0),(0,0,0,1)}$ yields a spread over $\mathbb{P}^3_{\mathbb{F}_q}$.
- Now let q be even. Then there is some $\psi \in \mathbb{F}_q$ such that $x^2 + x + \psi \in \mathbb{F}_q[x]$ is irreducible. Defining $L_{a,b} = \overline{(1,0,a,b),(0,1,\psi b,a+b)}$ and $L_\infty = \overline{(0,0,1,0),(0,0,0,1)}$ yields a spread over $\mathbb{P}^3_{\mathbb{F}_q}$.

Spreads constructed using this method will be known as “Hopf spreads.”

Gorla's Construction

According to Gorla, it is possible to construct a spread of k-dimensional hyperplanes inside $\mathbb{P}^n_{\mathbb{F}_q}$ if and only if $k+1$ divides $n+1$.
Let $p(x) \in \mathbb{F}_q[x]$ be irreducible, monic, and degree $k+1$, and let P be its companion matrix. Then one can construct a spread of spaces of the form

$$\operatorname{rowsp} \underbrace{\begin{pmatrix} 0 & \cdots & 0 & I_{k+1} & A_1 & \cdots & A_j \end{pmatrix}}_{\frac{n+1}{k+1}} : A_i \in \mathbb{F}_q[P].$$

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When $k=1$ and $n=3$, this construction is identical to the Hopf spread! But this also provides new examples of spreads.

Non-Hopf Spreads

Definition

A **regulus** is a set of mutually-skew lines $\mathcal{R}$ such that there is a quadric surface Q where $\bigcup_{R \in \mathcal{R}} R = Q$, like this.

Every regulus $\mathcal{R}$ admits an **opposite regulus** $\mathcal{R}^*$.

The Hopf spread contains reguli: for example $\mathcal{R} = \{L_{a,0} : a \in k\} \cup \{L_\infty\}$ is a regulus.

Given a spread $\mathcal{S}$ containing a regulus $\mathcal{R}$, the set of lines $(\mathcal{S} \setminus \mathcal{R}) \cup \mathcal{R}^*$ is also a spread.

It was once conjectured that every spread could be constructed by starting with the Hopf spread and replacing a succession of reguli with their opposites. This is now known to be false. In fact, there are spreads that contain no reguli whatsoever!

Maximal Partial Spreads

Note that a spread over $\mathbb{P}^3_{\mathbb{F}_q}$ comprises $q^2 + 1$ mutually-skew lines.

Definition

A **partial spread** of $\mathbb{P}^3_{\mathbb{F}_q}$ with **deficiency** d is a set of $q^2 + 1 - d$ mutually-skew lines. A **maximal partial spread** is a partial spread of positive deficiency that is not contained in any larger partial spread.

Maximal partial spreads are also instrumental to the study of geproci sets; in fact, given a maximal partial spread $\mathcal{M}$, the set $\mathbb{P}^3_{\mathbb{F}_q} \setminus \left(\bigcup_{L \in \mathcal{M}} L\right)$ is geproci.

Projecting a Line to a Line via... a Line

Definition

Given three lines $L_1, L_2, L_3 \subseteq \mathbb{P}^3_k$ where $L_1 \cap L_2 = L_2 \cap L_3 = \varnothing$, we can define the function $\pi(L_1, L_2, L_3) : L_1 \to L_3$ as follows: take $p \in L_1$. Then there is a unique line T such that $p \in T$, $T \cap L_2 \neq \varnothing$, and $T \cap L_3 \neq \varnothing$. Then define $\pi(L_1, L_2, L_3)(p) = T \cap L_3$. This is the **projection** of L_1 to L_3 **via** L_2.

Here is a demonstration.

Groupoids

Definition

A **groupoid** is a category $\mathcal{G}$ where every morphism is invertible.

- For any object $G \in \mathcal{G}$, $\text{Hom}_{\mathcal{G}}(G, G) = \text{Aut}_{\mathcal{G}}(G)$ is a group.

$\text{Aut}_{\mathcal{G}}(G)$ is a "group of the groupoid."

- Whenever $\text{Hom}_{\mathcal{G}}(G_1, G_2) \neq \varnothing$, then $\text{Aut}_{\mathcal{G}}(G_1) \cong \text{Aut}_{\mathcal{G}}(G_2)$.

So when $\text{Hom}_{\mathcal{G}}(G_1, G_2) \neq \varnothing$ for all $G_1, G_2 \in \mathcal{G}$, $\mathcal{G}$ induces only one group of the groupoid, up to isomorphism. Then it makes sense to say "the" group of the groupoid, $\text{Aut}_{\mathcal{G}}$.

Groupoids of Lines

Theorem

Let $\mathcal{L}$ be a set of lines in $\mathbb{P}^3_F$. Define Π to be the composition-closure of the set of functions $\{\pi(L_i, L_j, L_k) : L_i, L_j, L_k \in \mathcal{L}, L_i \cap L_j = L_j \cap L_k = \varnothing\}$. Then $(\mathcal{L}, \Pi)$ is a groupoid.

In this case, any group of the groupoid is a subgroup of $\mathrm{Aut}(\mathbb{P}^1_F) \cong \mathrm{PGL}(2, F)$.

What can we say about this groupoid and its corresponding group(s)? In characteristic 0, when is it finite versus infinite? When does $\mathrm{Aut}_{(\mathcal{L},\Pi)}(L)$ have finite orbits, or finitely many orbits?

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What can we say about this groupoid and its corresponding group(s)? In characteristic 0, when is it finite versus infinite? When does $\mathrm{Aut}_{(\mathcal{L},\Pi)}(L)$ have finite orbits, or finitely many orbits?

NOTE: If $\mathcal{L}$ contains lines L_i, L_j, L_k where $L_i \cap L_j \neq \varnothing$, then neither $\pi(L_i, L_j, L_k)$ nor $\pi(L_k, L_j, L_i)$ are defined. So when can we characterize whether $\mathrm{Hom}_{(\mathcal{L},\Pi)}(L_i, L_k) = \varnothing$? If $\mathcal{L} = \mathcal{R} \cup \mathcal{R}^*$, then $\mathrm{Hom}(R, R') = \varnothing$ for all $R \in \mathcal{R}$ and $R' \in \mathcal{R}^*$.

Ganger's Results

In her 2024 thesis, Ganger used the technique of **transversals** to prove the following theorem:

Theorem (Ganger Corollary 2.5)

The group of the groupoid for the Hopf spread induced by the degree-2 field extension $\mathbb{F}_{q^2}/\mathbb{F}_q$ *over a finite field is isomorphic to the quotient* $\mathbb{F}_{q^2}^*/\mathbb{F}_q^* \cong C_{q+1}$.

Definition

Given a set of lines $\mathcal{L}$ in $\mathbb{P}^3_F$, a **transversal** is a line T in $\mathbb{P}^3_F$ such that $T \cap \overline{L} \neq \varnothing$ for all $L \in \mathcal{L}$.

The Hopf spread has exactly two transversals T_1, T_2 for any finite field. The intersection of transversal with a line $L \in \mathcal{S}$ is a fixed point of $\mathrm{Aut}_{(\mathcal{S},\Pi)}(L)$!

Representing $\mathrm{Aut}_{\mathcal{G}} \leq \mathrm{PGL}(2, F)$

Let $U = \overline{u_0 u_1}, V = \overline{v_0 v_1}, W = \overline{w_0 w_1}$ be lines in $\mathbb{P}^3$. Then any point on U can be written as $au_0 + bu_1$ for $(a, b) \in \mathbb{P}^1$ and any point on W can be written as $cw_0 + dw_1$ for $(c, d) \in \mathbb{P}^1$. Then if

$$cw_0 + dw_1 = \pi(U, V, W)(au_0 + bu_1),$$

we must have $\overline{(au_0 + bu_1)(cw_0 + dw_1)} \cap V \neq \varnothing$.
Therefore the wedge product

$$(au_0 + bu_1) \wedge v_0 \wedge v_1 \wedge (cw_0 + dw_1) = 0.$$

We can write this as a matrix formula

$$\begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} u_0 \wedge v_0 \wedge v_1 \wedge w_0 & u_0 \wedge v_0 \wedge v_1 \wedge w_1 \\ u_1 \wedge v_0 \wedge v_1 \wedge w_0 & u_1 \wedge v_0 \wedge v_1 \wedge w_1 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = 0,$$

or rewrite this equivalently:

$$\begin{pmatrix} -u_0 \wedge v_0 \wedge v_1 \wedge w_1 & -u_1 \wedge v_0 \wedge v_1 \wedge w_1 \\ u_0 \wedge v_0 \wedge v_1 \wedge w_0 & u_1 \wedge v_0 \wedge v_1 \wedge w_0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix}.$$

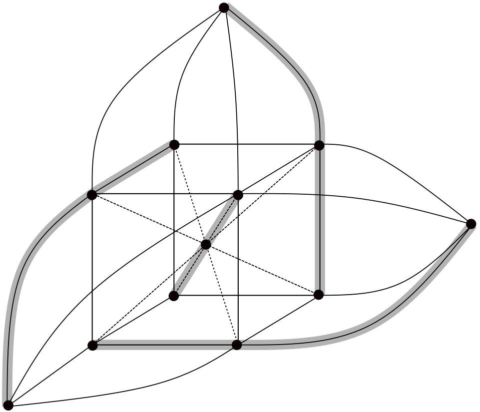
Representing $\text{Aut}_{\mathcal{G}} \leq \text{PGL}(2, k)$

The 2×2 matrix

$$\begin{pmatrix} -u_0 \wedge v_0 \wedge v_1 \wedge w_1 & -u_1 \wedge v_0 \wedge v_1 \wedge w_1 \\ u_0 \wedge v_0 \wedge v_1 \wedge w_0 & u_1 \wedge v_0 \wedge v_1 \wedge w_0 \end{pmatrix}$$

represents $\pi(U, V, W)$, given parametrizations
$U = \overline{u_0 u_1}$, $V = \overline{v_0 v_1}$, $W = \overline{w_0 w_1}$.
Note $r \wedge s \wedge t \wedge u = \det \begin{pmatrix} r & s & t & u \end{pmatrix}$.
This allows us to use computational methods to experiment on the group of the groupoid!

Example: The D_4 configuration in $\mathbb{P}^3_{\mathbb{C}}$



The D_4 configuration is a $(3, 4)$-geproci half-grid. It is a $(12_4, 16_3)$-configuration. What is the group of the groupoid of the 16 lines?

Example: The D_4 configuration

Theorem

Let $\mathcal{L}$ be the 16 lines of the D_4 configuration and let Π be the composition-closure of the projection functions. Then the group of the groupoid $\mathcal{G} = (\mathcal{L}, \Pi)$ is $Aut_{\mathcal{G}} \cong S_3$.

Argument boils down to:

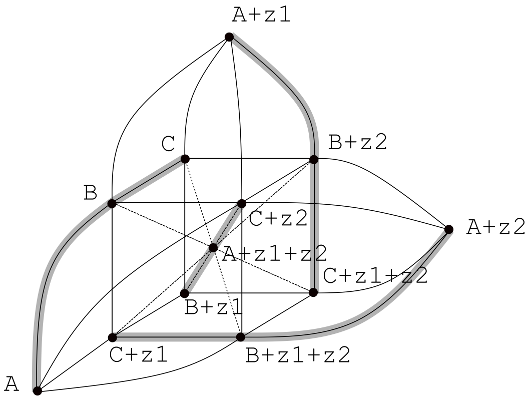
- $\text{Hom}_{\mathcal{G}}(L, L') \neq \varnothing$ for $L, L' \in \mathcal{L}$, so $\text{Aut}_{\mathcal{G}}$ is well-defined.
- Let $q \in L$ be a quadruple point and $\pi \in \text{Hom}_{\mathcal{G}}(L, L')$. Then $\pi(q)$ is a quadruple point. So $\text{Aut}_{\mathcal{G}} \leq S_3$.
- We have found automorphisms in $\text{Aut}_{\mathcal{G}}(L)$ of orders 2 and 3, so $\text{Aut}_{\mathcal{G}} \cong S_3$.

A Helpful Labeling

Let $\{A, B, C\}$ be a set of three letters, and consider the group $(\mathbb{Z}/2\mathbb{Z})^2 = \langle z_1, z_2 \rangle$. Then one can label the vertices of the D_4 configuration with the elements of $\{A, B, C\} \oplus (\mathbb{Z}/2\mathbb{Z})^2$ in such a way that there is a bijective correspondence between the lines of the D_4 and triples of the form
$\{A + g, B + g', C + g'' : g + g' + g'' = 0\} \subseteq \{A, B, C\} \oplus (\mathbb{Z}/2\mathbb{Z})^2$.

This makes the theorem on the previous slide easier to prove because you can divide the vertices into "types" A, B, and C.

A Helpful Labeling



- $\{A, B, C\}$
- $\{A + z_1, B + z_2, C + z_1 + z_2\}$
- $\{A + z_2, B + z_1 + z_2, C + z_1\}$
- $\{A + z_1 + z_2, B + z_1, C + z_2\}$