VECTOR BUNDLES AND PROJECTIVE MODULES

BY

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Serre [9, §50] has shown that there is a one-to-one correspondence between algebraic vector bundles over an affine variety and finitely generated projective modules over its coordinate ring. For some time, it has been assumed that a similar correspondence exists between topological vector bundles over a compact Hausdorff space X and finitely generated projective modules over the ring of continuous real-valued functions on X. A number of examples of projective modules have been given using this correspondence. However, no rigorous treatment of the correspondence seems to have been given. I will give such a treatment here and then give some of the examples which may be constructed in this way.

1. Preliminaries. Let K denote either the real numbers, complex numbers or quaternions. A K-vector bundle ξ over a topological space X consists of a space $E(\xi)$ (the total space), a continuous map $p : E(\xi) \to X$ (the projection) which is onto, and, on each fiber $F_x(\xi) = p^{-1}(x)$, the structure of a finite dimensional vector space over K. These objects are required to satisfy the following condition: for each $x \in X$, there is a neighborhood U of x, an integer n, and a homeomorphism $\phi : p^{-1}(U) \to U \times K^n$ such that on each fiber ϕ is a K-homomorphism. The fibers $u \times K^n$ of $U \times K^n$ are K-vector spaces in the obvious way. Note that I do not require n to be a constant. The dimension of the fiber F_x may vary with x. However, this dimension is clearly locally constant and so will be constant if X is connected.

A subbundle of ξ is, by definition, a subset $E_1 \subset E(\xi)$ such that $E_1 \cap F_x$ is a K-subspace of F_x for each x and such that E_1 with the projection $p \mid E_1$ and the K-structure on its fibers induced by that of E forms a K-vector bundle over X.

A map of K-vector bundles $f : \xi \to \eta$ is defined to be a continuous map $f : E(\xi) \to E(\eta)$ such that $pf = p$ and such that $f \mid F_x(\xi) : F_x(\xi) \to F_x(\eta)$ is a K-homomorphism. It is clear that the K-vector bundles over X and their maps form an additive category.

A section s of ξ over a subset $A \subset X$ is a continuous map $s : A \to E(\xi)$ such that $ps(x) = x$. It follows immediately from the definition of a vector bundle that for any $x \in X$, there is a neighborhood U of x and sections $s_1, \ldots, s_n$ of ξ

over U such that $s_1(y), ..., s_n(y)$ form a K-base for F_y for each $y \in U$. I will say that $s_1, ..., s_n$ form a local base at x. Any section of ξ over U can be written as $s(y) = a_1(y)s_1(y) + ... + a_n(y)s_n(y)$ where $a_i(y) \in K$. Note that s is continuous if and only if each function a_i is. (This is immediate for the local base $e_1, ..., e_n$ which we get from the definition of a vector bundle. If $s_1, ..., s_n$ is another local base, $s_i(y) = \sum_j a_{ij}(y)e_j(y)$ and $y \to (a_{ij}(y))$ is a continuous map $U \to GL(n,K)$. The result then follows from the fact that $A \to A^{-1}$ is a continuous map in $GL(n,K)$.) Similarly, if $s_1, ..., s_m$ is a local base for ξ at x, $t_1, ..., t_n$ is a local base for η at x, and $f: \xi \to \eta$, then near x, $f(s_i(y)) = \sum_j a_{ij}(y)t_j(y)$ and f is continuous if and only if each $a_{ij}(y)$ is continuous. If $f: \xi \to \eta$ is one-to-one and onto, the fact that $A \to A^{-1}$ in $GL(n, K)$ is continuous shows that f^{-1} is continuous. In other words, such an f must be an isomorphism.

LEMMA 1. *Let* $t_1, ..., t_k$ *be sections of* ξ *over a neighborhood* U *of* x *such that* $t_1(x), ..., t_k(x)$ *are linearly independent. Then there is a neighborhood* V *of* x *such that* $t_1(y), ..., t_k(y)$ *are linearly independent for each* $y \in V$.

Proof. Let $s_1, ..., s_n$ be a local base at x. Let $t_i(y) = \sum a_{ij}(y)s_j(y)$. Some $k \times k$ submatrix of $(a_{ij}(x))$ must be nonsingular by hypothesis. Therefore this same submatrix must be nonsingular in $(a_{ij}(y))$ for all y sufficiently near x. For the real and complex numbers, this follows by taking determinants. For the quaternions, we must first replace the matrix by a real $4k \times 4k$ one. We then know that the nonvanishing of this real determinant is equivalent to the non-singularity of the original matrix. The existence of the nonsingular $k \times k$ sub-matrix clearly implies the conclusion of the lemma.

In general, it is not true that a map of vector bundles has a kernel and image in the category of vector bundles. For example, let $X = I$, the unit interval, and let ξ be the product bundle $I \times K$ where $p(x,y) = x$. Let $f: \xi \to \xi$ by $f(x,y) = (x,xy)$. Then the image of f has a fiber of dimension 1 everywhere except at $x = 0$ where the fiber is zero. Thus im f cannot be a vector bundle. Similarly, ker f cannot be a vector bundle. However, this is the only sort of thing which can go wrong.

PROPOSITION 1. *Let* $f: \xi \to \eta$ *be a map of vector bundles. Then the following statements are equivalent*:
   (1)   im f *is a subbundle of* η;
   (2)   ker f *is a subbundle of* ξ;
   (3)   *the dimensions of the fibers of* im f *are locally constant*;
   (4)   *the dimensions of the fibers of* ker f *are locally constant*.

Proof. It is clear that (3) and (4) are equivalent and are implied by either (1) or (2). To see that (3) implies (1), let $x \in X$, choose a local base $s_1, ..., s_m$ for ξ at x and a local base $t_1, ..., t_n$ for η at x. Let k be the dimension of the fiber of im f at x. By renumbering if necessary, we can assume that $fs_1(x), ..., fs_k(x)$ span $F_x(\text{im } f)$ and so are linearly independent. By renumbering again, we can assume

$fs_1(x), ..., fs_k(x), \ t_{k+1}(x), ..., t_n(x)$ are linearly independent. Thus, by Lemma 1 and the local constancy of the dimension of the fiber of $\eta, fs_1, ..., fs_k, t_{k+1}, ..., t_n$ form a local base for η at x. By the hypothesis and Lemma 1, $fs_1, ..., fs_k$ form a local base for im$f$ at x. It is now clear that im f is a subbundle of η.

To see that (3) implies (2), let $s_1, ..., s_m$ be as above. For all y near x, we can write $fs_i(y) = \sum_1^k a_{ij}(y)fs_j(y)$ for $i > k$. Let $s_i'(y) = s_i(y) - \sum_1^k a_{ij}(y)s_j(y)$. Then $s_{k+1}', ..., s_m'$ are local sections of ker f and are linearly independent near x. Since there are exactly the right number of them, they form a local base for ker f. It follows as above that ker f is a subbundle of ξ.

REMARK 1. Without any hypothesis, this proof shows that if $\dim_k F_x(\text{im} f) = n$, then $\dim_k F_y(\text{im } f) \geqq n$ for all y in some neighborhood of x.

DEFINITION. An inner product on a K-vector space V is a function $(,)$: $V \times V \to K$ such that (x, y) is K-linear in x for each y, $(y, x) = (x, y)^-$, and $(x, x) > 0$ unless $x = 0$.

An inner product on a K-vector bundle ξ is given by an inner product $(,)_x$ on each fiber F_x which varies continuously with x. In other words, if Y is the subset of $E(\xi) \times E(\xi)$ consisting of all pairs (e_1, e_2) with $p(e_1) = p(e_2)$, the totality of the $(,)_x$ must give a continuous function $Y \to K$.

LEMMA 2. *If X is paracompact, every K-vector bundle over X has an inner product.*

Proof (Milnor [7, V, Theorem 5]). Let $\{U_\alpha\}$ be a locally finite covering of X such that $p^{-1}(U_\alpha) = U_\alpha \times K^{n_\alpha}$. It is trivial to construct an inner product $(,)_{\alpha,x}$ on each $p^{-1}(U_\alpha)$. Let $\{\omega_\alpha\}$ be a real partition of unity on X for the covering $\{U_\alpha\}$. Define $(e_1, e_2)_x = \sum_\alpha \omega_\alpha(x)(e_1, e_2)_{\alpha,x}$.

DEFINITION. If ξ, η are vector bundles, the direct sum $\xi \oplus \eta$ is defined by letting $E(\xi \oplus \eta)$ be the subset of $E(\xi) \times E(\eta)$ consisting of pairs (e_1, e_2) with $p(e_1) = p(e_2)$. The projection $p : E(\xi \oplus \eta) \to X$ is defined by $p(e_1, e_2) = p(e_1) = p(e_2)$. Clearly $F_x(\xi \oplus \eta) = F_x(\xi) \times F_x(\eta)$. We give this a K-module structure in the obvious way. It is clear that $\xi \oplus \eta$ is a vector bundle.

PROPOSITION 2. *If X is paracompact, any subbundle η of a vector bundle ξ is a direct summand.*

Proof (Milnor [7]). Choose an inner product for ξ. This inner product defines a projection $f_x : F_x(\xi) \to F_x(\eta)$ which varies continuously with x. Thus $f : \xi \to \eta$ is a map of vector bundles. Let $\zeta = \text{ker} f$. By Proposition 1, ζ is a subbundle of ξ. Clearly $\eta \oplus \zeta = \xi$.

2. **Modules of sections.** Let $C(X) = C_K(X)$ be the ring of continuous K-valued functions on X. If ξ is a K-vector bundle over X let $\Gamma(\xi)$ be the set of all sections of ξ over X. If $s_1, s_2 \in \Gamma(\xi)$, define $(s_1 + s_2)(x) = s_1(x) + s_2(x)$. If $s \in \Gamma(\xi)$ and $a \in C(X)$, define $(as)(x) = a(x)s(x)$. With these definitions, $\Gamma(\xi)$ becomes a $C(X)$-

module. Clearly Γ is an additive functor from the category of K-vector bundles over X to the category of $C(X)$-modules. If ξ is the trivial bundle $E(\xi) = X \times K^n$, then $\Gamma(\xi)$ is obviously a free $C(X)$-module on n generators.

The object of this section is to show that if X is normal, Γ gives an isomorphism $\text{Hom}(\xi, \eta) \approx \text{Hom}_{C(X)}(\Gamma(\xi), \Gamma(\eta))$.

LEMMA 3. *Let X be normal. Let U be a neighborhood of x, and let s be a section of a vector bundle ξ over U. Then there is a section s' of ξ over X such that s' and s agree in some neighborhood of x.*

Proof. Let V, W be neighborhoods of x such that $\bar{V} \subset U$, $\bar{W} \subset V$. Let ω be a real-valued function on X such that $\omega | \bar{W} = 1$, $\omega | X - V = 0$. Let $s'(y) = \omega(y)s(y)$ if $y \in U$ and $s'(y) = 0$ if $y \notin U$.

COROLLARY 1. *Let X be normal. For any $x \in X$ there are elements $s_1, ..., s_n \in \Gamma(\xi)$ which form a local base at x.*

COROLLARY 2. *Let X be normal. If $f, g : \xi \to \eta$ and $\Gamma(f) = \Gamma(g): \Gamma(\xi) \to \Gamma(\eta)$, then $f = g$.*

Proof. Given $e \in E(\xi)$, with $p(e) = x$, there is a section s over a neighborhood U of x with $s(x) = e$. By Lemma 3, there is a section $s' \in \Gamma(\xi)$ with $s'(x) = e$. Now $f(e) = fs'(x) = (\Gamma(f)s')(x) = (\Gamma(g)s')(x) = g(e)$.

LEMMA 4. *Let X be normal. Let $s \in \Gamma(\xi)$. Suppose $s(x) = 0$. Then there are elements $s_1, ..., s_k \in \Gamma(\xi)$, $a_i, ..., a_k \in C(X)$ such that $a_i(x) = 0$ for $i = 1, ...,k$ and $s = \sum a_i s_i$.*

Proof. Let $s_1, ..., s_n \in \Gamma(\xi)$ be a local base at x (Corollary 1). Let $s(y) = \sum b_i(y)s_i(y)$ near x, $b_i(y) \in K$. Let $a_i \in C(X)$ be such that a_i and b_i agree in a neighborhood of x. These exist by Lemma 3 applied to $X \times K$. Then $s' = s - \sum a_i s_i$ vanishes in a neighborhood U of x. Let V be a neighborhood of x such that $\bar{V} \subset U$. Let $a \in C(X)$ be zero at x and 1 on $X - V$. Then $s = as' + \sum a_i s_i$. But $a(x) = 0$ and $a_i(x) = b_i(x) = 0$.

COROLLARY 3. *Let I_x be the (2-sided) ideal of $C(X)$ consisting of all $a \in C(X)$ with $a(x) = 0$. Then $\Gamma(\xi)/I_x\Gamma(\xi) \approx F_x(\xi)$, the isomorphism being given by $s \to s(x)$.*

This follows from Lemma 4 and the proof of Corollary 2.

THEOREM 1. *Let X be normal. Given any $C(X)$-map $F: \Gamma(\xi) \to \Gamma(\eta)$, there is a unique K-bundle map $f: \xi \to \eta$ such that $F = \Gamma(f)$.*

Proof. The uniqueness follows from Corollary 2. Now, F induces a map $f_x: \Gamma(\xi)/I_x\Gamma(\xi) \to \Gamma(\eta)/I_x\Gamma(\eta)$. The totality of these yield a map $f: E(\xi) \to E(\eta)$. This map is K-linear on fibers. Its continuity remains to be verified. If $s \in \Gamma(\xi)$, $(fs)(x) = f_x s(x) = (F(s))(x)$ by construction so that $F = \Gamma(f)$.

To check that f is continuous, let $s_1, ..., s_m \in \Gamma(\xi)$ be a local base at x. If $e \in E(\xi)$ and $p(e)$ is near x, we have $e = \sum a_i(e)s_i(p(e))$ where the a_i are continuous K-valued functions. Now, $f(e) = \sum a_i(e)fs_i(p(e))$. Since $fs_i = F(s_i)$, fs_i is a continuous section of η. Now all terms in the sum are continuous in e. Thus f is continuous.

COROLLARY 4. *Let X be normal and ξ, η two K-vector bundles over X. Then $\xi \approx \eta$ if and only if $\Gamma(\xi) \approx \Gamma(\eta)$ as $C(X)$-modules.*

3. **Projective modules.** I will now show that if X is compact Hausdorff, the $C(X)$-modules which can occur as $\Gamma(\xi)$ for some ξ are exactly the finitely generated projective modules.

LEMMA 5. *Let X be compact Hausdorff. Let ξ be any K-vector bundle over X. Then there is a trivial vector bundle ζ (i.e., $E(\zeta) = X \times K^n$) and an epimorphism $f: \zeta \to \xi$.*

Proof. For each $x \in X$, choose a set of sections $s_{x,1}, ..., s_{x,k_x} \in \Gamma(\xi)$ which form a local base over some neighborhood U_x of x. A finite number of the U_x cover X. Therefore, there are a finite number of sections $s_1, ..., s_n \in \Gamma(\xi)$ such that $s_1(x), ..., s_n(x)$ span $F_x(\xi)$ for every x. Let ζ be the trivial bundle with $E(\zeta) = X \times K^n$. Then $\Gamma(\zeta)$ is a free $C(X)$-module on n generators $e_1, ..., e_n$. Map $\Gamma(\zeta) \to \Gamma(\xi)$ by $e_i \to s_i$. By Theorem 1, this is induced by a map $f: \zeta \to \xi$. Since $fe_i = s_i$, $s_i(x) \in \text{im } f$. Therefore f is onto.

COROLLARY 5. *If X is compact Hausdorff, any K-vector bundle over X is a direct summand of a trivial K-vector bundle ζ.*

Proof. Let $f: \zeta \to \xi$ as in Lemma 5. Let $\eta = \ker f$. By Proposition 1, η is a subbundle of ζ. By Proposition 2, $\zeta = \eta \oplus \xi'$. Clearly $\xi' \approx \xi$.

COROLLARY 6. *If X is compact Hausdorff and ξ is any K-vector bundle over X, then $\Gamma(\xi)$ is a finitely generated projective $C(X)$-module.*

Proof. By Corollary 5, $\Gamma(\xi)$ is a direct summand of $\Gamma(\zeta)$ which is a finitely generated free $C(X)$-module.

THEOREM 2. *Let X be compact Hausdorff. Then a $C(X)$-module P is isomorphic to a module of the form $\Gamma(\xi)$ if and only if P is finitely generated and projective.*

Proof. The "only if" follows from Corollary 6. Suppose now that P is finitely generated and projective. Then P is a direct summand of a finitely generated free $C(X)$-module F. Therefore, there is an idempotent endomorphism $g: F \to F$ with $P \approx \text{im } g$. Now $F = \Gamma(\zeta)$ where ζ is a trivial K-vector bundle. By Theorem 1, $g = \Gamma(f)$ where $f: \zeta \to \zeta$. Since $g^2 = g$, Theorem 1 implies $f^2 = f$. If we knew that $\xi = \text{im } f$ was a subbundle of ζ, we would have (Proposition 2) $\zeta = \xi \oplus \eta$ where $\eta = \ker f$, and so $P \approx \text{im } \Gamma(f) = \Gamma(\xi)$ since Γ is an additive functor. By

Proposition 1, it is enough to show that $\dim_K F_x(\xi)$ is locally constant. Since $f^2 = f$, $\eta = \ker f = \operatorname{im}(1 - f)$ and $F_x(\zeta) = F_x(\xi) \oplus F_x(\eta)$. Suppose $\dim_K F_x(\xi) = h$, $\dim_K F_x(\eta) = k$. By Remark 1 applied to f and $1 - f$ respectively, we have $\dim F_y(\xi) \geqq h$, $\dim F_y(\eta) \geqq k$ for all y in some neighborhood of x. But

$$\dim F_y(\xi) + \dim F_y(\eta) = \dim F_y(\zeta) = h + k$$

is a constant. Thus $\dim F_x(\zeta)$ is locally constant.

4. **Examples.** The rings $C(X)$ are generally too large to give nice examples. We would like examples of projective modules over affine rings. Therefore, we proceed as follows: find some affine ring $\Lambda \subset C(X)$ and a finitely generated projective Λ-module P. Then $C(X) \otimes_\Lambda P$ is a finitely generated projective $C(X)$-module and so is isomorphic to some $\Gamma(\xi)$. To prove that P is nontrivial in some sense (not free, indecomposable, etc.) it will then suffice to show that ξ is nontrivial in the same sense (by Corollary 4).

EXAMPLE 1 (KAPLANSKY). *A projective module with a free complement which is not free.*

Let τ^n be the tangent bundle of the n-sphere S^n. This is, of course, a real vector bundle. The usual imbedding $S^n \subset E^{n+1}$ shows that $\tau^n \oplus v^1$ is trivial where v^1 is the normal bundle [7, I]. Clearly v^1 is also trivial. Thus $\Gamma(\tau^n)$ is projective and has a free complement. In fact, $\Gamma(\tau_n) \oplus C(X)$ is free. However, by Corollary 4, $\Gamma(\tau^n)$ cannot be free unless S^n is parallelizable. This can only happen if $n = 0, 1, 3, 7$ [4; 5]. Of course, we do not need to use this deep result. It would suffice to take n even. In this case, $\Gamma(\tau^n)$ is even indecomposable. To see this, suppose $\tau^n = \xi \oplus \eta$. Then the Euler classes satisfy $X(\tau^n) = X(\xi)X(\eta)$ [7, VIII, Theorem 13]. But, if ξ, $\eta \neq 0$, $X(\xi)$ and $X(\eta)$ have dimensions between 1 and $n-1$ and so must be 0. On the other hand, $X(\tau^n) = \chi(S^n)\mu = 2\mu$ where μ generates $H^n(S^n)$ [7, IX, Theorem 16].

We must now reduce $\Gamma(\tau^n)$ to a reasonable size. The bundle $\zeta = \tau^n \oplus v^1$ is trivial. An element of $E(\zeta)$ consists of a point $x = (x_0, \ldots, x_n) \in S^n$ and a vector $u = (u_0, \ldots, u_n)$ of E^{n+1}. The subbundle v^1 consists of those pairs $[x, u]$ for which $u = \lambda x$ for some real λ. We can project ζ onto v^1 by $f([x, u]) = [x, \lambda x]$ where $\lambda = (x, u)$ is the inner product $\sum x_i u_i$. Let $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$, the 1 occurring in the ith place. Then $s_i : x \to [x, e_i]$, $i = 0, \ldots, n$, form a set of generators for $\Gamma(\zeta)$. Now $f s_i : x \to [x, (x, e_i)x] = [x, x_i x] = \sum_j x_i x_j s_j(x)$ so $f s_i = x_i \sum_j x_j s_j$. Thus $\Gamma(f)$ and so $\Gamma(\tau^n)$, can be defined using only polynomials. Let $\Lambda = R[x_0, \ldots, x_n]/(x_0^2 + x_1^2 + \ldots + x_n^2 - 1)$, R being the real numbers. Then $\Lambda \subset C(S^n)$. Let F be a free Λ-module on $n + 1$ generators $s_0, \ldots, s_n$. Define $g : F \to F$ by $g(s_i) = \sum_j x_i x_j s_j$. Then g is idempotent, so $P = \ker g$ is projective. Clearly $C(X) \otimes_\Lambda P \approx \Gamma(\tau^n)$. Thus P is indecomposable if n is even and P is not free for $n \neq 0, 1, 3, 7$. However, $P \oplus \operatorname{im} g = F$ is free. Now $\operatorname{im} g$ contains $g(\sum x_i s_i) = \sum x_j s_j$ and this clearly generates $\operatorname{im} g$. Thus $\operatorname{im} g$ is free so P has a free comple-

ment (i.e., its direct sum with a finitely generated free module is free). Since $P \approx F/\text{im } g$, we can express the result as follows.

THEOREM 3. *Let* $\Lambda = R[x_0, ..., x_n]/(x_0^2 + ... + x_n^2 - 1)$, *$R$ being the real numbers, $n \geqq 1$. Let P be the Λ-module with generators $s_0, ..., s_n$ and relation* $\sum x_i s_i = 0$. *Then* $P \oplus \Lambda$ *is free but P is not free for $n \neq 1, 3, 7$. If n is even, P is even indecomposable.*

Another way to express this result [10, §9] is to observe that $(x_0, ..., x_n)$ is a unimodular row over Λ. The above results show that it cannot be the first row of a unimodular Λ-matrix unless $n = 1, 3, 7$. Equivalently, $(x_0, ..., x_n)$ cannot be transformed into $(1, 0, ..., 0)$ by a unimodular Λ-matrix unless $n = 1, 3, 7$.

REMARK. It is known that the complexification of the tangent bundle of S^n is a trivial bundle. This naturally suggests the question: Is CP free over $C\Lambda$, C being the complex numbers? The answer is "yes" as is shown by the following series of unimodular transformations. Let $2u = x_0 + ix_1$, $2v = x_0 - ix_1$. Then $(x_0, x_1, x_2, ..., x_n) = (u + v, -iu + iv, x_2, ..., x_n) \to (2u, -iu + iv, x_2, ..., x_n) \to (2u, iv, x_2, ..., x_n) \to (2u + (1 - 2u)(-4iu(iv) + x_2^2 + ... + x_n^2), iv, x_2, ..., x_n) = (1, iv, x_2, ..., x_n) \to (1, 0, ..., 0)$. These transformations, applied to the base for CF yield a new base in terms of which $(x_0, ..., x_n)$ becomes $(1, 0, ..., 0)$. Clearly factoring out the submodule generated by this element gives a free module.

REMARK. It is interesting to note that P is indeed free for $n = 1, 3, 7$. We can choose bases $e_0, ..., e_n$ for the complex numbers, quaternions, and Cayley numbers such that $e_0 = 1$ and the norm of $x = \sum x_i e_i$ is $Nx = \sum x_i^2$. Consider the matrix R_x of the linear transformation $y \to yx$. The entries of R_x are integral linear combinations of the x_i. The first row of R_x is $(x_0, ..., x_n)$. The determinant of R_x is Nx, Nx^2, Nx^4 for $n = 1, 3, 7$. Consequently, if we regard R_x as a matrix over Λ, it is unimodular and completes $(x_0, ..., x_n)$.

EXAMPLE 2 (CHASE, see [3, Remark, p. 451]). *A projective module with a large number of generators.*

There is a standard real line bundle γ^1 over real projective n-space defined as follows [7, I]: P^n is the set of all lines l through the origin in E^{n+1}. $E(\gamma^1)$ consists of all pairs (l, x) where $x \in l$. The projection is $p(l, x) = l$. The real vector space structure on the fibers is the obvious one. Let σ^{k-1} be a trivial real vector bundle over P^n with fibers of dimension $k-1$ (i.e., a $(k-1)$-plane bundle). Then $\xi = \gamma^1 \oplus \sigma^{k-1}$ is a k-plane bundle over P. I claim that $\Gamma(\xi)$ cannot be generated over $C(X)$ by fewer than $n+k$ elements. If it could, it would be a direct summand of a free $C(X)$-module on $n+k-1$ generators. Therefore, ξ would be a direct summand of a trivial $(n + k - 1)$-plane bundle. In other words, there would be an $(n-1)$-plane bundle η over P^n such that $\xi \oplus \eta$ is trivial. Now, the total Stiefel-Whitney class of ξ is $W(\xi) = W(\gamma^1) = 1 + \alpha$ where α generates $H^1(P^n, Z_2)$. Let $W(\eta) = 1 + a_1 \alpha + ... + \alpha^r$ where r is the largest integer such that $W_r(\eta) \neq 0$. Since η is an $(n-1)$-plane bundle, $r \leqq n - 1$. By the Whitney product theorem [7, II],

$W(\xi)W(\eta) = W(\xi \oplus \eta) = 1$ since $\xi \oplus \eta$ is trivial. But, $W(\xi)W(\eta) = \alpha^{r+1} +$ lower powers of α. Since $r \leqq n - 1$, $r + 1 \leqq n$ and $\alpha^{r+1} \neq 0$. Thus we have arrived at a contradiction.

If P is a projective module over a commutative ring Λ, the rank of P is defined to be the maximum of $\dim_{\Lambda/M}(P/MP)$ over all maximal ideals M of Λ. If $\Lambda = C(X)$ for $K = R$ or C, X compact Hausdorff, every maximal ideal of Λ has the form I_x [6, Chapter IV, 19C]. Therefore, by Corollary 3, the rank of $\Gamma(\xi)$ is just the maximum of the dimensions of the fibers of ξ. Therefore, we can restate the result proved above by saying that there is a finitely generated module of rank k over $C(P^n)$ which cannot be generated by fewer than $n + k$ elements.

Now $\Gamma(\xi)$ is the direct sum of $\Gamma(\gamma^1)$ and a free module. In order to replace $C(P^n)$ by a suitable affine ring Λ, we must find a more explicit description of $\Gamma(\gamma^1)$. Let $p: S^n \to P^n$ be the natural projection. Then the map $C(P) \to C(S^n)$ by $f \to fp$ is a monomorphism. Therefore, we can identify $C(P^n)$ with the subring of $C(S^n)$ consisting of those functions which satisfy $g(-x) = g(x)$. If s is a section of γ^1, then $sp(x) = ((x), h(x)x)$ where (x) denotes the line through 0 and x, and $h(x) \in R$. Now $sp(x) = sp(-x) = ((x), h(-x)(-x))$ so $h(-x) = -h(x)$. Conversely, such an h gives a section s of γ^1. Thus $\Gamma(\gamma^1)$ is isomorphic to the submodule M of $C(S^n)$ consisting of those functions h satisfying $h(-x) = -h(x)$. Consider the functions $h_i(x) = x_i$. These give sections $s_0, \ldots, s_n$ of γ^1. For any $x \in S^n$, some $s_i(x) \neq 0$. Since γ^1 is a line bundle, the $s_i(x)$ generate $F_x(\gamma^1)$ for every x. Therefore $s_0, \ldots, s_n$ generate $\Gamma(\gamma^1)$ or, equivalently, $x_0, \ldots, x_n$ generate M.

We can write M as a direct summand of a free module as follows: Let F be the free $C(X)$-module on $n + 1$ generators $e_0, \ldots, e_n$. Map $F \to M$ by $e_i \to x_i$. Map $M \to F$ by $h \to x_i h e_0 + \ldots + x_i h e_n$. Note that $x_i h \in C(P^n)$ if $h \in M$. Clearly $M \to F \to M$ is the identity.

Now, choose $\Lambda \subset C(P^n)$ to be the subring of $R[x_0, \ldots, x_n]/(x_0^2 + \ldots + x_n^2 - 1)$ consisting of all those polynomials all of whose terms have even (total) degree. Let P be the Λ-submodule of $R[x_0, \ldots, x_n]/(x_0^2 + \ldots + x_n^2 - 1)$ generated by $x_0, \ldots, x_n$. Thus P consists of those polynomials all of whose terms have odd (total) degree. Now P is a direct summand of a free module by the same construction as that given for M in the preceding paragraph. Therefore P is projective. Clearly $C(P^n) \otimes_\Lambda P \approx M$. If P_k is the direct sum of P with a free Λ-module on $k - 1$ generators, then $C(P^n) \otimes_\Lambda P_k \approx \Gamma(\xi)$. Therefore, P_k cannot be generated by fewer than $n + k$ elements. Since P_k has rank k, we have proved the following result.

THEOREM 4. *Let Λ be the subring of $R[x_0, \ldots, x_n]/(x_0^2 + \ldots + x^2 - 1)$ consisting of all polynomials all of whose terms have even degree. Then for every integer $k \geqq 1$ there is a finitely generated projective Λ-module P_k of rank k which cannot be generated by fewer than $n + k$ elements.*

Note that Λ is an affine ring of dimension n.

REMARK.   Suppose X is a finite complex of dimension n. Let ξ be a real k-plane bundle over X. Then standard obstruction theory arguments show that ξ is induced from the standard k-plane bundle γ^k over $G_{k,n}$ (the Grassmannian) by a map $X \to G_{k,n}$. Now γ^k has a complementary n-plane bundle γ''' such that $\gamma^k \oplus \gamma'''$ is trivial. Let η over X be induced by γ'''. Then $\xi \oplus \eta$ is a trivial $(n + k)$-plane bundle. It follows that $\Gamma(\xi)$ can be generated by $n + k$ elements. The example considered above shows that this is the best possible result. This suggests the following problem, due to Serre.

PROBLEM.   Let Λ be a commutative noetherian ring whose maximal ideal spectrum has dimension n [10, §2; 3]. Let P be a finitely generated projective Λ-module of rank k. Can P be generated by $n + k$ elements?

By Theorem 4, $n + k$ would be the best possible value. Bass [3] has given an upper bound for the number of generators of P. For $k = 1$, this bound is $n + 1$, so the answer to the problem in case $k = 1$ is "yes."

EXAMPLE 3. UNIQUE FACTORIZATION. If Λ is an integral domain with quotient field F, and P is a finitely generated projective Λ-module, the rank of P is the dimension over F of $F \otimes_\Lambda P$. This can easily be seen by localization [10, §3]. Therefore, if we express P as a direct summand of a free module M, the matrix E which gives a retraction of M on P will have rank equal to the rank of P.

If Λ is a unique factorization domain (UFD), it is well known that every projective Λ-module of rank 1 is free. This is easily seen as follows: Over a field, a rank 1 matrix has all its rows linearly dependent and so can be written in the form $(a_i b_j)$. Consequently, if Λ is any integral domain, any rank 1 matrix with entries in Λ has the form $(a_i b_j d^{-1})$ with $a_i, b_j, d \in \Lambda$. Now, if Λ is a UFD, any prime p dividing d must also divide all $a_i b_j$ since the entries are in Λ. Therefore, either p divides all a_i or p divides all b_j. If, say, p divides all a_i, replace a_i by $p^{-1} a_i$ and d by $p^{-1} d$. Continue this process as long as there are primes dividing d. In this way we see that any rank 1 matrix over a UFD Λ has the form $(a_i b_j)$ with $a_i, b_j \in \Lambda$. If $E = (a_i b_j)$ is idempotent and nonzero, $\sum a_i b_j = 1$. Consequently, the image of E is generated by the column (a_i) and so is a free Λ-module.

If Λ is a regular domain [1, §4], every rank 1 prime ideal is projective. This is equivalent to the fact that regular local rings are UFD's [2]. Therefore, in this case, Λ is a UFD if and only if every rank 1 projective module is free.

Consider the rings $\Lambda_n = R[x_0, \ldots, x_n]/(x_0^2 + \ldots + x_n^2 - 1)$ and $C\Lambda_n = C[x_0, \ldots, x_n]/(x_0^2 + \ldots + x_n^2 - 1)$. These are easily seen to be regular if $n > 0$. Our experience in Example 1 suggests that we can get information about projective modules over these rings by looking at real and complex vector bundles over S^n.

If $\Lambda \subset C(X)$, and P is a finitely generated projective Λ-module, then

$$C(X) \otimes_\Lambda P \approx \Gamma(\xi)$$

for some ξ. Now

$$F_x(\xi) = \Gamma(\xi)/I_x\Gamma(\xi) = (C(X) \otimes_\Lambda P)/(I_x \otimes_\Lambda P) = (C(X)/I_x) \otimes_\Lambda P.$$

Now assuming $K \subset \Lambda$, we see that $M_x = I_x \cap \Lambda$ is a maximal ideal of Λ and $\Lambda/M_x = C(X)/I_x \approx K$. Therefore, $F_x(\xi) \approx P/M_x P$. Thus, the rank of P is the maximum of the dimensions of the fibers of ξ. In particular, projective modules of rank 1 correspond to line bundles over X.

Now, there are nontrivial real line bundles over S^n for $n > 1$ and no nontrivial complex line bundles over S^n for $n = 1$ or $n \geq 3$. There is, however, a nontrivial bundle (Möbius band) over S^1 and a nontrivial complex line bundle over S^2 (the tangent bundle of S^2 with its usual complex structure). This leads us to conjecture the following theorem, which turns out to be correct.

THEOREM 5[2]. *For $n \geq 1$, Λ_n is a UFD for $n \neq 1$ and $C\Lambda_n$ is a UFD for $n \neq 2$. Furthermore, Λ_1 and $C\Lambda_2$ are not UFD's.*

Of course, we cannot prove that Λ_n is a UFD by topological means. Many projective modules may conceivably become free in passing from Λ_n to $C(S^n)$. However, the result is easily proved by methods of Nagata [8]. Let $\Gamma_n = R[y_0, \ldots, y_n, t]/(y_0^2 + \ldots + y_n^2 - t^2)$. Then $\Gamma_n[t^{-1}] \approx \Lambda_n[t, t^{-1}]$, by sending y_i into tx_i. If we can show Γ_n is a UFD, it will follow that $\Lambda_n[t, t^{-1}]$ is a UFD. By [8, Lemma 1], $\Lambda_n[t]$ is then a UFD and so Λ_n is a UFD. Let $u = y_n + t$, $v = y_n - t$. Then $\Gamma_n = R[y_0, \ldots, y_{n-1}, u, v]/(y_0^2 + \ldots + y_{n-1}^2 + uv)$. Therefore, $\Gamma_n[u^{-1}] = R[y_0, \ldots, y_{n-1}, u, u^{-1}]$ is a UFD. By [8, Lemma 1], to show Γ_n is a UFD we need only show that (u) is a prime ideal in Γ_n. This is clear for $n \geq 2$ because $\Gamma_n/(u) = R[y_0, \ldots, y_{n-1}, v]/(y_0^2 + \ldots + y_{n-1}^2)$. For $C\Gamma_n$, this argument only works if $n \geq 3$ because $y_0^2 + y_1^2$ is not irreducible over C. The fact that $C\Lambda_1$ is a UFD was shown by Nagata [8] by the same argument applied directly to $C\Lambda_1$ using $u = x_0 + ix_1$, $v = x_0 - ix_1$.

The fact that Λ_1 is not a UFD was also shown by Nagata [8] by elementary means. However, I will deduce it here from the nontriviality of the Möbius bundle. If we identify S^1 with the set of real numbers $\theta \mod 2\pi$, this bundle can be defined as the subbundle μ of $S^1 \times R^2$ consisting of all $(\theta, \lambda \cos \theta/2, \lambda \sin \theta/2)$ with $\lambda \in R$. Clearly, the fiber undergoes a half twist as θ goes from 0 to 2π. The retraction of $S^1 \times R^2$ on μ is given by $\rho : (\theta, u, v) \to (\theta, \lambda \cos \theta/2, \lambda \sin \theta/2)$ where $\lambda = u \cos \theta/2 + v \sin \theta/2$. Note that while λ depends on $\theta \mod 4\pi$, $\lambda \cos \theta/2$ and $\lambda \sin \theta/2$ only depend on $\theta \mod 2\pi$. The usual representation of S^1 by $x^2 + y^2 = 1$ is related to the one above by $x = \cos \theta$, $y = \sin \theta$. In terms of x and y, the retraction becomes $\rho : (x, y; u, v) \to (x, y; 1/2(1 + x)u + 1/2(yv), 1/2(yu) + 1/2(1 - x)v)$. Let P be the rank 1 projective Λ_1-module which is the image of the retraction $\rho : (u, v) \to (1/2(1 + x)u + 1/2(yv), 1/2(yu) + 1/2(1 - x)v)$ defined on the free Λ_1-module $\{(u, v) \mid u, v \in \Lambda_1\}$. Then $C(S^1) \otimes_{\Lambda_1} P = \Gamma(\mu)$. This is not free since the Möbius bundle is nontrivial.

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(2) The first statement in this theorem is due to P. Samuel (unpublished). The statements about Λ_1 and $C\Lambda_1$ are due to Nagata [8].

To show that $C\Lambda_2$ is not a UFD, we first find a simple way to describe the complex structure on the tangent bundle τ of S^2. Identify S^2 with the set of quaternions x such that $|x| = 1$, $x \perp 1$. A tangent vector to S^2 at x is a quaternion y such that $y \perp 1$, $y \perp x$. Therefore, $E(\tau)$ may be identified with the set of pairs of quaternions (x,y) satisfying $|x| = 1$, $x \perp 1$, $y \perp 1$, $y \perp x$. The projection is $p(x,y) = x$. Now, if $x \in S^2$, $x^2 = -1$. Therefore $y \perp 1$, $y \perp x$ implies $xy \perp x$ and $xy \perp x^2 = -1$. Thus $J:(x,y) \to (x,xy)$ sends $E(\tau)$ into itself and preserves fibers. Since $J^2 = -1$, we can identify the action of J on τ with multiplication by $i = (-1)^{1/2}$. This makes τ a complex line bundle τ^1.

Now, as in Example 1, $\Gamma(\tau)$ can be regarded as the submodule of the free $C(S^2)$-module $\{(y_1,y_2,y_3) \mid y_i \in C(S^2)\}$ consisting of those (y_1,y_2,y_3) such that $x_1y_1 + x_2y_2 + x_3y_3 = 0$. The map $J:\tau \to \tau$ induces a map $J:\Gamma(\tau) \to \Gamma(\tau)$ given by $J(y_1,y_2,y_3) = (x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1)$ these being the components of the corresponding quaternion product xy. The projective Λ_2-module P considered in Example 1 is the submodule consisting of all (y_1,y_2,y_3) with $y_i \in \Lambda_2$. This is clearly stable under J. Therefore, we can make P into a module P' over $C\Lambda_2$ by identifying J with multiplication by $i = (-1)^{1/2}$. Clearly

$$C(S^2) \otimes_{C\Lambda_2} P' = \Gamma(\tau^1).$$

To show that P' is projective over $C\Lambda_2$, note that we could have made P into a module P'' over $C\Lambda_2$ by identifying J with multiplication by $-i$ instead of i. Now $C \otimes P \approx P' \oplus P''$ as $C\Lambda_2$-modules, the isomorphism $C \otimes P \to P' \oplus P''$ being given by $1 \otimes p \to (p,p)$, $i \otimes p \to (Jp, -Jp)$. Since P is Λ_2-projective, $C \otimes P$ is $C\Lambda_2$-projective, so P' and P'' are also $C\Lambda_2$-projective. Since τ is nontrivial, so is τ^1. Therefore, P' is the required nonfree projective $C\Lambda_2$-module of rank 1. Thus we have shown that $C\Lambda_2$ is not a UFD.

REMARK. The almost complex structure on S^6 can be defined exactly as above using the Cayley numbers instead of the quaternions. There is no need to worry about nonassociativity since all the above calculations can be done in the subalgebra generated by x and y which is associative. The third Chern class of the complex tangent bundle is equal to its Euler class [7, XI] which is 2 times the generator of $H^6(S^6)$. Therefore, the bundle and hence the resulting projective $C\Lambda_6$-module P' has no free complement. However, if we consider P' as a Λ_6-module only, it has a free complement as in Example 1. Therefore, the module $P' \oplus C\Lambda_6$ has no free complement as $C\Lambda_6$-module but is free when regarded as a Λ_6-module. The same comment applies to P' over $C\Lambda_2$ but $C\Lambda_6$ gives a somewhat more interesting example because Λ_6 and $C\Lambda_6$ are regular UFD's and so are among the nicest possible affine algebras.

EXAMPLE 4. EXTENDING THE GROUNDFIELD. The following question was raised by J. Towber. It is in some sense dual to the one answered by the above remark.

QUESTION. Let Λ be an affine algebra over a field K. Let P be a finitely generated

projective Λ-module. Let L be an extension field of K. Suppose LP is $L\Lambda$-free. What can be said about P?

At first sight, one might hope that P itself would be free, at least if Λ is nice enough. However, the first remark in Example 1 shows that this is not the case. The rings involved, Λ_n and $C\Lambda_n$, are even regular UFD's for $n \geqq 3$. In this example, P has a free complement. Therefore, one might hope that, in general, the module P in the above question would have a free complement. However, even this turns out to be false. In this section Λ_n, $C\Lambda_n$ have the same meaning as in Example 3.

THEOREM 6.   *There is a finitely generated projective module A over Λ_4 such that A has no free complement but CA is free over $C\Lambda_4$.*

The vector bundle used to prove this theorem is the one associated with the Hopf fibering $S^7 \to S^4$. Let V be the 2-dimensional quaternionic vector space consisting of pairs of quaternions (q_1, q_2). The quaternionic projective line QP^1 is the set of (quaternionic) lines through 0 in V. Since such a line is determined by a single point on it (other than 0), QP^1 is obtained from $V - 0$ by making the identifications $(q_1, q_2) \sim (\lambda q_1, \lambda q_2)$ for every nonzero quaternion λ. A representative (q_1, q_2) of a point in QP^1 can obviously be normalized to satisfy $|q_1|^2 + |q_2|^2 = 1$.

There is a standard quaternionic line bundle γ^1 over QP^1 analogous to the real line bundle over P^n considered in Example 2. The total space $E(\gamma^1)$ consists of all pairs (l, y) where l is a line through 0 in V and $y \in l$. The projection is $p(l, y) = l$. If we identify l by a point $q = (q_1, q_2)$ on it, any other point on it has the form $y = (y_1, y_2) = (\lambda q_1, \lambda q_2) = \lambda q$. Therefore, we can alternatively define $E(\gamma^1)$ as the set of all $(q, \lambda q)$ where $q \in V - 0$, λ is any quaternion, and where $(q, \lambda q)$ is identified with $(\mu q, \lambda q)$ for $\mu \neq 0$.

The bundle γ^1 is a subbundle of the trivial bundle consisting of all (q, y) where $q \neq 0$ is subjected to the same identification as before and $y \in V$ is independent of q. If we always choose q to be normalized, the projection p of this bundle on γ^1 can be defined by $(q, y) \to (q, \lambda q)$ where $\lambda = \langle y, q \rangle = y_1 \bar{q}_1 + y_2 \bar{q}_2$. This is easily seen to be independent of the normalized representative q.

It is now necessary to identify QP^1 with S^4 in a nice way. If (q_1, q_2) represents a point of QP^1 and $q_1 \neq 0$, then $(q_1, q_2) \sim (1, q)$ where $q = q_1^{-1} q_2$. If $q_1 = 0$, $(q_1, q_2) \sim (0, 1)$. In this way, QP^1 is identified with the set of quaternions plus a point at infinity corresponding to $(0, 1)$. Now, S^4 is the set of points in E^5 satisfying $x_0^2 + x_1^2 + \ldots + x_4^2 = 1$. Identify the hyperplane $x_0 = 0$ with the quaternions by $q = x_1 + ix_2 + jx_3 + kx_4$. Let e_0 be the point $(1, 0, 0, 0, 0)$. By stereographic projection from e_0 on the hyperplane $x_0 = 0$, we can map $S^4 - e_0$ homeomorphically onto the quaternions. By elementary analytic geometry, the formula for this map is found to be $x \to q$ where

$$q = \frac{x_1 + ix_2 + jx_3 + kx_4}{1 - x_0}.$$

In this way, we can map $S^4 - e_0 \to QP^1 - (0,1)$ by $x \to (1, q)$. Since S^4, QP^1 are the one-point compactifications of $S^4 - e_0, QP^1 - (0,1)$ respectively, this map can be extended to a homeomorphism $S^4 \to QP^1$ by sending e_0 to $(0,1)$.

The representative $(1, q)$ is not normalized. Since $x_0^2 + \ldots + x_5^2 = 1$, it is easy to check that the normalized form is $x \to q = (q_1, q_2)$ where

$$(q_1, q_2) = \left(\left(\frac{1 - x_0}{2} \right)^{1/2}, \frac{x_1 + ix_2 + jx_3 + kx_4}{(2(1 - x_0))^{1/2}} \right).$$

For $x_0 = 0, q_1 = 0$ and q_2 is indeterminate. However, $|q_2| \to 1$ as $x_0 \to 0$.

The trivial bundle considered above now has the form $S^4 \times V$ and the projection ρ of this bundle on γ^1 is given by $\rho(x, y) = (x, \lambda q)$ where $q = (q_1, q_2)$ is the normalized expression given in the preceding paragraph and $\lambda = \langle y, q \rangle$. Substituting the value of q in terms of x now gives

$$\rho(x, y_1, y_2) = (x, \tfrac{1}{2}y_1(1 - x_0) + \tfrac{1}{2}y_2(x_1 - ix_2 - jx_3 - kx_4),$$

$$\tfrac{1}{2}y_1(x_1 + ix_2 + jx_3 + kx_4) + \tfrac{1}{2}y_2(1 + x_0)).$$

Let K denote the quaternions. Let $M = \{(y_1, y_2) \,|\, y_1, y_2 \in K\Lambda_4\}$ be a free module on two generators over $K\Lambda_4$. Let $\rho: M \to M$ be defined by

$$\rho(y_1, y_2) = (\tfrac{1}{2}y_1(1 - x_0) + \tfrac{1}{2}y_2(x_1 - ix_2 - jx_3 - kx_4),$$

$$\tfrac{1}{2}y_1(x_1 + ix_2 + jx_3 + kx_4) + \tfrac{1}{2}y_2(1 + x_0)).$$

Then ρ is an idempotent endomorphism of M. Its image P is thus projective and clearly $C_K(S^4) \otimes_{K\Lambda_4} P \approx \Gamma(\gamma^1)$.

Let $f: M \to M$ by $f(y_1, y_2) = (-y_2, y_1)$. An easy calculation shows that $f^{-1}(1 - \rho)f$ sends (y_1, y_2) into

$$(\tfrac{1}{2}y_1(1 - x_0) + \tfrac{1}{2}y_2(x_1 + ix_2 + jx_3 + kx_4),$$

$$\tfrac{1}{2}y_1(x_1 - ix_2 - jx_3 - kx_4) + \tfrac{1}{2}y_2(1 + x_0)).$$

Therefore, ker $\rho = \text{im }(1 - \rho) \approx \text{im } f^{-1}(1 - \rho)f \approx P'$ where P' is the conjugate $K\Lambda_4$ module obtained from P by letting i, j, k act as $-i, -j, -k$. Since $P \oplus \ker \rho$ is free, this shows that $P \oplus P'$ is free.

Now, let A be P considered as a module over Λ_4. Then, as in Example 3, $C \otimes A \approx P \oplus P'$ as modules over $C\Lambda_4$. Thus $CA = C \otimes A$ is free over $C\Lambda_4$.

However, $C_R(S^4) \otimes_{\Lambda_4} A \approx \Gamma(\gamma^1)$ where γ^1 is now considered as a real vector bundle. The associated sphere bundle is just the Hopf bundle $S^7 \to S^4$ with fiber S^3 [11, Part 2, §20.4]. Consideration of the spectral sequence (or even the homotopy sequence) of this bundle shows that the Euler class $X(\gamma^1)$ generates

$H^4(S^4)$. The Stiefel-Whitney class $W_4(\gamma^1)$ is $X(\gamma^1)$ reduced mod 2 [7, VIII] and therefore, generates $H^4(S^4, Z_2)$. If θ is any trivial real vector bundle over S^4, $W_4(\theta \oplus \gamma^1) = W_4(\gamma^1) \neq 0$. Thus $\theta + \gamma^1$ can never be trivial. Consequently, A has no free complement.

A similar example over Λ_2 could be given by using the Hopf bundle $S^3 \to S^2$. However, $C\Lambda_4$ is somewhat nicer than $C\Lambda_2$, being a UFD.

REMARK. After this paper was submitted, the paper *Sur les anneaux factoriels*, Bull. Soc. Math. France **89** (1961), 155–173 by P. Samuel appeared. This contains his proof of Theorem 5. Theorem 3 is also stated (for the case $n = 2$) and the connection with topology is pointed out but no details are given.

I have also recently become acquainted with the mimeographed notes *Differential topology*, Princeton, 1958, by J. Milnor. These notes contain (Lemma 2.19, Theorem 2.20) an analogue of Lemma 5 in which X is assumed paracompact and finite dimensional and the dimension of the fiber is assumed constant. The proof in fact only requires X to be paracompact and have the property that every covering has a finite dimensional refinement. Consequently, Theorem 2 also holds for such spaces X provided we consider only bundles whose fibers have bounded dimension. This will always be the case if X has a finite number of components.

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