Frank left off talking about the canonical divisor of a variety.

The sheaf of relative differentials $\Omega_{X/k}$ of X.

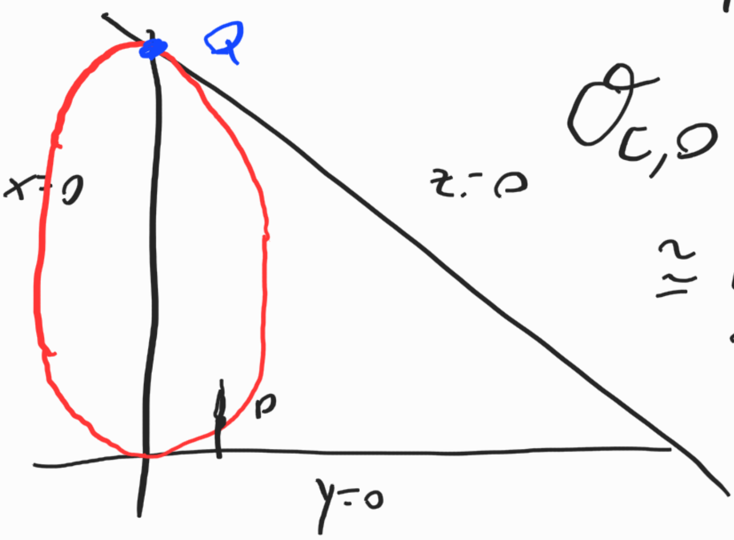
The canonical bundle $\underline{\omega_X} := \underline{\bigwedge^{\dim X} \Omega_{X/k}}$.

The canonical divisor $K_X \in Cl(X)$ satisfies $\mathcal{O}_X(K_X) = \omega_X$.

$df_1 \wedge df_2 \wedge \cdots \wedge df_n$ ← f_i are regular functions

$n = \dim X$.

Let C be the parabola $yz - x^2 = 0$



$$\mathcal{O}_{C,Q} = \left(k[x,y,z]/(yz-x^2)\right)_{(Q),0}$$

$$\cong k\left(\frac{x}{z}, \frac{y}{z}\right)/\left(\frac{y}{z} - \frac{x^2}{z^2}\right)$$ localize

degree

$$\cong k\left(\frac{x}{z}\right) = k(\alpha).$$

Example: Compute $\mathrm{div}(d\alpha)$

$\mathrm{div}(df) = \sum_{P \in X} \mathrm{ord}_P\left(\frac{df}{dt_P}\right)$ where

t_P is the generator for $m_P \subseteq \mathcal{O}_{C,P}$.

Compute $\mathrm{ord}_P\left(\frac{d\alpha}{dt_P}\right)$ for P off the line $z=0$. $P=(a,b,1)$

$t_P = \alpha - a$. $\frac{d\alpha}{d(\alpha-a)} = \frac{d\alpha}{d\alpha - da} = 1$.

$\mathrm{ord}_P(1) = 0$.

For Q, $\mathrm{ord}_Q\left(\frac{d\alpha}{dt_Q}\right) = -2$.

$\mathrm{div}(d\alpha) = -2Q$.

Adjunction Formula: $K_C = (\underline{K_X} + C)|_C$.

$K_{\mathbb{P}^2} = -3 \cdot L$ where L is a line in the plane.

$C \sim 2 \cdot L$. $K_C = (-3L + 2L)|_C$

$= -L|_C = -P_1 - P_2$ (P_1 & P_2 might be the same).

Elliptic Curves.

G is a cubic.

$K_G = (K_{\mathbb{P}^2} + G)|_G$

$(-3L + 3L)|_G = 0$

$\mathcal{O}_G(K_G)(X) = \mathcal{O}_G(0)(X) = \underline{k}.$

$\ell(D) = \dim \underbrace{\mathcal{O}_X(D)(X)}_{\mathcal{L}(D)}.$

$\boxed{\ell(K_G) = 1.}$

Rieman-Roch.

$\ell(D) - \ell(K-D) = \deg(D) - g + 1.$

$g = \ell(K)$

$\ell(K) - \ell(0) = \deg K - g + 1$

g " 1 "

$$g-1 = \deg K - g + 1$$

$$\deg(K) = 2g-2.$$

Let's take E to be a smooth genus-1 curve. Let $P \in E$ be a point on E.

$\ell(mP)$ for $m > 0$.

$$\ell(mP) + \underbrace{\ell(K-mP)}_{=0} = \underbrace{\deg(mP)}_{=m} \cancel{-g+1}$$

$$\deg(K-mP) = -m < 0 \leftarrow$$

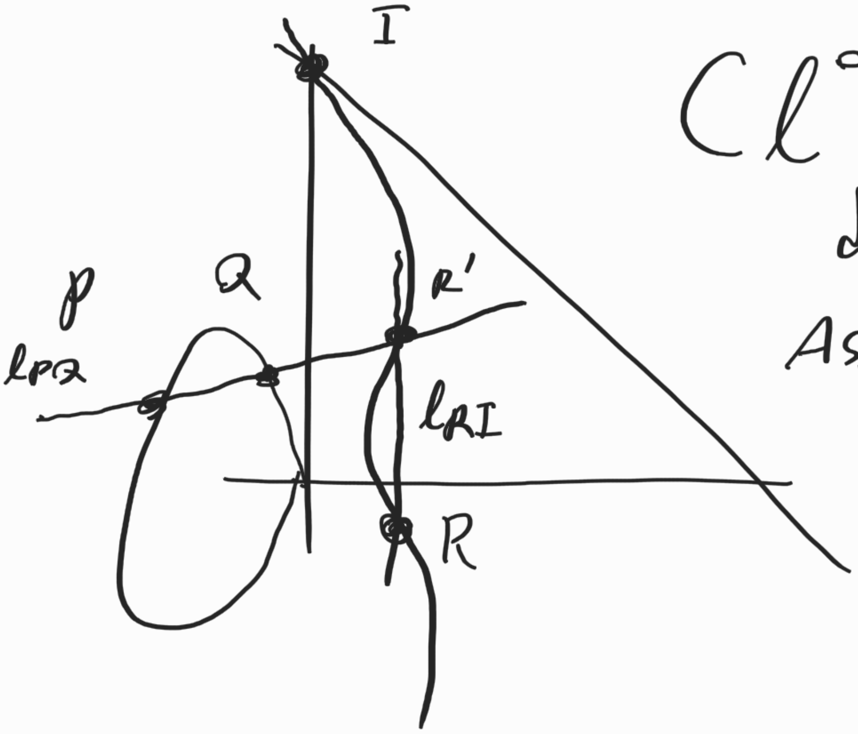
So $\ell(K-mP) = 0$

$$\ell(mP) = m.$$

m	$\mathcal{O}(mP)$ generators
0	1
1	1
2	$1, x$
3	$1, x, y$
4	$1, x, y, x^2$
5	$1, x, y, x^2, xy$
$\boxed{6}$	$\underbrace{1, x, y, x^2, xy, y^2, x^3}_{7}$

$$\boxed{a + bx + cy + dx^2 + exy + fy^2 + gx^3 = 0}$$

RR shows any smooth genus-1 curve is cubic.



$Cl^\circ(E)$ is the degree-0 divisors.

Associate P to $P - I$ for all $P \in E$.

$(P - I) + (Q - I) \overset{?}{\sim} R - I$.

$P + Q - 2I \overset{?}{\sim} R - I$

$\underline{P + Q - R - I \text{ principal?}}$ $\quad P + Q - R - I \sim 0$.

$$\operatorname{div}\left(\frac{l_{PQ}}{l_{RI}}\right) = P + Q + R' - (\cancel{R} + I + \cancel{R'})$$

Linear Systems of divisors

Let D be a divisor on a curve X. Then the complete linear system $|D|$ is the set of divisors in $\operatorname{Div}(X)$ that are linearly equivalent to D.

We can make a association between

$\mathcal{O}_X(D)$ and $|D|$.

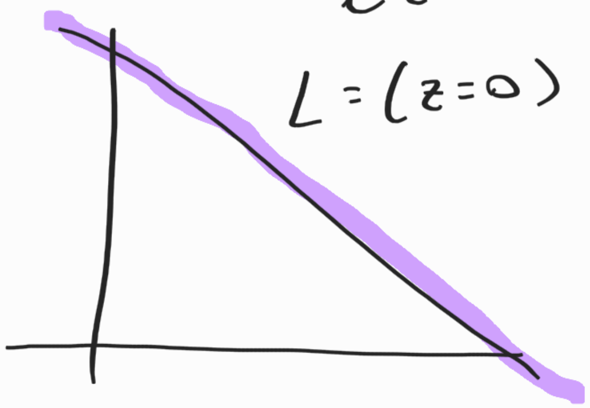
$$f \longmapsto \underline{\mathrm{div}(f)} + D \sim D.$$

This association doesn't care about scalar multiples. So $\dim |D| = \ell(D) - 1$.

If we look at $\mathbb{P}^2$, we can take the complete linear system $|3L|$, the class of all cubic curves in $\mathbb{P}^2$.

Denote $\underline{|3L - P_1|}$ as the class of cubics with a <u>base point</u> at P_1. ie all cubics that go through P_1.

what is $\ell(3L)$?



$\mathcal{O}(3L)(x)$ will consist of regular functions generated by

$$1, \frac{x}{z}, \frac{y}{z}, \frac{x^2}{z^2}, \frac{xy}{z^2}, \frac{y^2}{z^2}, \frac{x^3}{z^3}, \frac{x^2y}{z^3}, \frac{xy^2}{z^3}, \frac{y^3}{z^3}.$$

$$\ell(3L) = 10$$

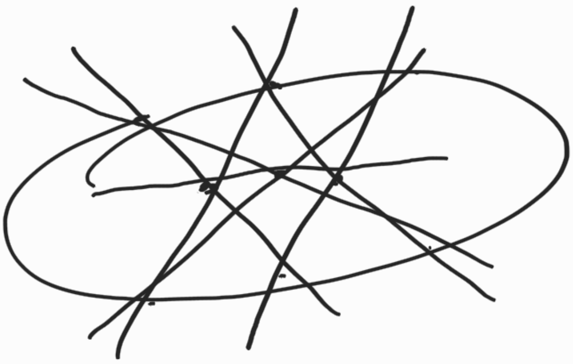
$$\dim |3L| = 9.$$

$$\dim |\underbrace{3L - P_1 - P_2 - \cdots - P_8}_{\Lambda}| = \underline{1}.$$

$C, C' \in \Lambda$. $\quad C \cap C' = \{P_1, \dots, P_8, \underline{\underline{P_9}}\}$.

<u>Cayley - Bacharach.</u>

For any 8 <u>general</u> points on the plane, there is a 9^{th} point P_9 such that any cubic that goes through the first 8 must go through P_9.



Pascal's Theorem