Differential Forms

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$\Phi[X]$

Recall we've seen $\mathsf{d}_x : \mathfrak{m}_x/\mathfrak{m}_x^2 \to \Theta_x^*$ is an isomorphism of vector spaces.

Consider $\Phi[X] = \left\{ \varphi : X \to \coprod_{x \in X} \Theta_x^* : \varphi(x) \in \Theta_x^* \right\}$.

For every regular function f, there is a linear form $\mathsf{d}_x f$ on Θ_x. Thus we have $\mathsf{d}f \in \Phi[X]$ satisfying $(\mathsf{d}f)(x) = \mathsf{d}_x f$.

Regular Differential 1-forms

An element $\varphi \in \Phi[X]$ is a *regular differential form* if each $x \in X$ has a neighborhood U such that $\varphi|_U$ in the $k[U]$-module $\Phi[U]$ is generated by elements of the form $\mathsf{d}f$, with $f \in k[U]$.

That is, there are $f_1, \ldots, f_m, g_1, \ldots, g_m \in k[U]$ such that

$$\varphi|_U = \sum_{i=1}^{m} g_i \mathsf{d} f_i. \tag{1}$$

We denote the $k[X]$-module of all regular differential forms on X as $\Omega[X]$.

Rules of Derivation

The map $\mathsf{d} : k[X] \to \Omega[X]$ follows these two rules:

$$\mathsf{d}(f+g) = \mathsf{d}f + \mathsf{d}g \qquad \mathsf{d}(fg) = f\mathsf{d}g + g\mathsf{d}f. \tag{2}$$

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Therefore, for $F \in k[X_1, \ldots, X_n]$ and $x_1, \ldots, x_n \in k[X]$, we have

$$\mathsf{d}(F(x_1, \ldots, x_n)) = \sum_{i=1}^{n} F_{X_i}(x_1, \ldots, x_n)\mathsf{d}x_i. \tag{3}$$

To prove this, one can use (2) to prove $\mathsf{d}(x^m) = mx^{m-1}\mathsf{d}x$ by induction on m. Then one can induct on n.

Example: $\Omega[\mathbb{A}^n]$

Since the differentials $\mathsf{d}_x t_1, \ldots, \mathsf{d}_x t_n$ form a basis on the vector space Θ_x^* for any $x \in \mathbb{A}^n$, any element $\varphi \in \Phi[\mathbb{A}^n]$ can be written uniquely of the form $\varphi = \sum_{i=1}^n \psi_i \mathsf{d} t_i$ for k-valued functions ψ_i.

If $\varphi \in \Omega[\mathbb{A}^n]$, then φ has a form of $\sum_{i=1}^m g_i \mathsf{d} f_i$ in a neighborhood of any x. Applying rule (3) to f_i, we get $\varphi = \sum_{i=1}^n h_i \mathsf{d} t_i$ for $h_i \in \mathcal{O}_x$. Since this expression is unique, the ψ_i must be regular at every $x \in X$ and so $\psi_i \in k[\mathbb{A}^n]$. Thus

$$\Omega[\mathbb{A}^n] = \bigoplus k[\mathbb{A}^n] \mathsf{d} t_i.$$

Canonical Divisor of a Plane Curve

Let $C \subseteq \mathbb{P}^2$ be a plane curve, and let $\omega \in \Omega[C]$. We will define $\text{ord}_x(\omega)$ as follows:

Let $t \in \mathcal{O}_x$ be a uniformizing parameter ($\text{ord}_x(t) = 1$). So $\{\mathsf{d}_x t\}$ is a basis for Θ_x^*. There is an $f \in k(C)$ such that $\omega = f\mathsf{d}t$. Then $\text{ord}_x(\omega) := \text{ord}_x(f)$.

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Can be rewritten $\mathrm{ord}_x(\omega) = -\mathrm{ord}_x(\mathsf{d}t/\omega)$.

Define $\mathrm{div}(\omega) = \sum_{x \in X} \mathrm{ord}_x(\omega) x$.

Example: $C = V(YZ - X^2)$

Let $C = V(YZ - X^2) \subseteq \mathbb{P}^2$. Then $k(C) = k(x, y)/(y - x^2)$ where $x = X/Z$ and $y = Y/Z$. Let $\omega = \mathsf{d}x$.

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Let $P = (a, b, c)$ where $c \neq 0$. Then $\mathcal{O}_P$ has uniformizing parameter $cx - a$. So $\mathrm{ord}_P(\omega) = -\mathrm{ord}_P(\mathsf{d}(cx - a)/\mathsf{d}x) = -\mathrm{ord}_P(c) = 0$.

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Let $Q = (0, 1, 0)$. Then $\mathcal{O}_Q$ has uniformizing parameter $x/y = X/Y$ since $\text{ord}_Q(X/Y) = 1 - 0 = 1$. Note $x/y = 1/x$ and

$$\frac{\mathsf{d}(1/x)}{\mathsf{d}x} = -\frac{\mathsf{d}x}{x^2\mathsf{d}x} = -1/x^2.$$

Then $\text{ord}_Q(\omega) = -\text{ord}_Q(-(1/x)^2) = -2$. Thus $\text{div}(\omega) = -2Q$.

Exterior Powers

Let V be a k-vector space. Define the r^{th} tensor power to be $T^rV = V^{\otimes r}$. Then denote $TV = \bigoplus_{i=1}^n T^iV$ and $\Lambda V = TV/(v \otimes v)$.

Let $\{v_1, \ldots, v_n\}$ be a basis for V. The r^{th} exterior power $\Lambda^r V$ is the k-vector space generated by exterior products of the form $v_{i_1} \wedge \cdots \wedge v_{i_r}$ with $1 \leq i_1 < \cdots < i_r \leq n$. Thus, $\dim_k \Lambda^r V = \binom{n}{r}$.

- For $\sigma \in S_r$, $v_1 \wedge \cdots \wedge v_r = \operatorname{sgn}(\sigma)\left(v_{\sigma(1)} \wedge \cdots \wedge v_{\sigma(r)}\right)$.
- For $\alpha \in \Lambda^a V$ and $\beta \in \Lambda^b V$, $\alpha \wedge \beta = (-1)^{ab} \beta \wedge \alpha \in \Lambda^{a+b} V$.

Regular Differential r-forms

Denote $\Phi^r[X] = \{ f : X \to \coprod_{x \in X} \Lambda^r \Theta_x^* : f(x) \in \Lambda^r \Theta_x^* \}$.

The exterior product on the $\Phi^r[X]$ sets is given by

$$(\omega_r \wedge \omega_s)(x) = \omega_r(x) \wedge \omega_s(x).$$

An element $\varphi \in \Phi^r[X]$ is a regular differential r-form if any point $x \in X$ has a neighborhood U such that $\varphi|_U$ is generated by $\mathsf{d}f_1 \wedge \cdots \wedge \mathsf{d}f_r$. All regular differential r-forms on X form a $k[X]$-module $\Omega^r[X]$.

Local Freeness

Theorem: Any nonsingular point x of an n-dimensional variety X has a neighborhood U such that $\Omega^r[U]$ is a free $k[U]$-module of rank $\binom{n}{r}$.

For any local parameters $u_1, \ldots, u_n$ on x, we can write $\omega \in \Omega^n[U]$ as

$$\omega = g\mathsf{d}u_1 \wedge \cdots \wedge \mathsf{d}u_n.$$

Canonical Divisor

Fix $x \in X$. For any two bases $\{\mathsf{d}_x u_1, \ldots, \mathsf{d}_x u_n\}$ and $\{\mathsf{d}_x v_1, \ldots, \mathsf{d}_x v_n\}$ on Θ_x^*, there exist $h_{ij} \in k[U]$ such that

$$\mathsf{d}u_i = \sum_{j=1}^{n} h_{ij} \mathsf{d}v_j.$$

We have $\det |h_{ij}(x)| =: J\left(\frac{u_1, \ldots, u_n}{v_1, \ldots, v_n}\right)(x) \neq 0$. This is called the *Jacobian*.

Thus for $\omega = g\mathsf{d}u_1 \wedge \cdots \wedge \mathsf{d}u_n$, $\operatorname{ord}_x(\omega) := \operatorname{ord}_x(g)$ is well-defined. Therefore $\operatorname{div}(\omega) = \sum_{x \in X} \operatorname{ord}_x(\omega)$ is well-defined.

Exterior Derivative d

We have seen $\mathsf{d} : k[X] \to \Omega^1[X]$. Now consider $\mathsf{d} : \Omega^1[X] \to \Omega^2[X]$ given by

$$\mathsf{d}(f\mathsf{d}t) = \mathsf{d}f \wedge \mathsf{d}t.$$

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Indeed, we can define $\mathsf{d} : \Omega^r[X] \to \Omega^{r+1}[X]$ as

$$\mathsf{d}(f\mathsf{d}u_1 \wedge \cdots \wedge \mathsf{d}u_r) = \mathsf{d}f \wedge \mathsf{d}u_1 \wedge \cdots \wedge \mathsf{d}u_r.$$

Note under this definition

$$\mathsf{d}(\mathsf{d}u_1 \wedge \cdots \wedge \mathsf{d}u_r) = \mathsf{d}1 \wedge \mathsf{d}u_1 \wedge \cdots \wedge \mathsf{d}u_r = 0.$$

Thus $\mathsf{d}^2 = 0$. We also get $\mathsf{d}(\alpha \wedge \beta) = \mathsf{d}\alpha \wedge \beta + (-1)^r(\alpha \wedge \mathsf{d}\beta)$ where α is an r-form.

De Rham Cohomology

The de Rham complex is the cochain complex

$$0 \to k(X) \xrightarrow{\mathsf{d}} \Omega^1(X) \xrightarrow{\mathsf{d}} \Omega^2(X) \to \cdots .$$

The i^{th} de Rham cohomology of X $H^i_{\text{dR}}(X) = \ker \mathsf{d}^i / \text{im } \mathsf{d}^{i-1}$.

For example $H^0_{\text{dR}}(X) \cong k$ is the space of constant functions on X.

Integration

Let $\dim X = n$, $\omega = f\,\mathsf{d}u_1 \wedge \cdots \wedge \mathsf{d}u_n$, and let U be open. Then

$$\int_U \omega = \int_U f(u_1, \ldots, u_n)\mathsf{d}u_1 \cdots \mathsf{d}u_n.$$