Automorphism Groups of Curves and Surfaces

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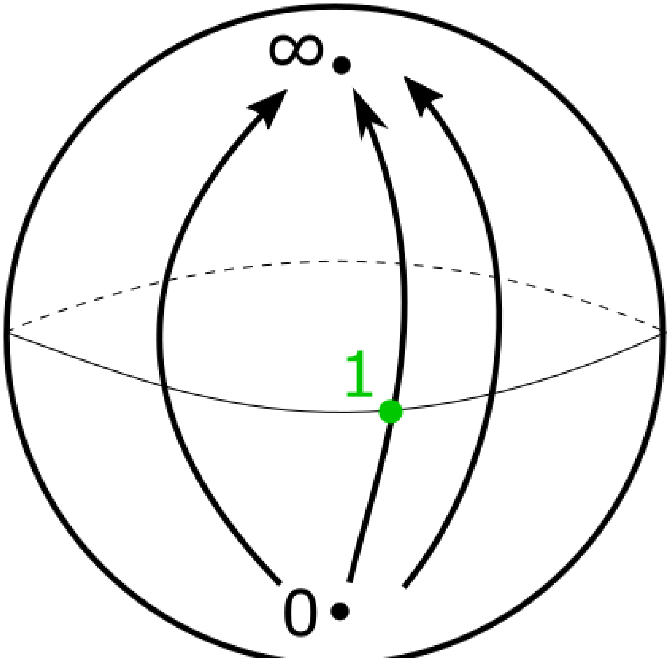
$\mathbb{P}^1_{\mathbb{C}}$

The group of automorphisms of $\mathbb{P}^1_{\mathbb{C}}$ is $\mathrm{PGL}(1, \mathbb{C}) = \mathrm{GL}(2, \mathbb{C})/\mathbb{C}^* I_2$. An automorphism of $\mathbb{P}^1_{\mathbb{C}}$ is called a *Möbius transformation*.

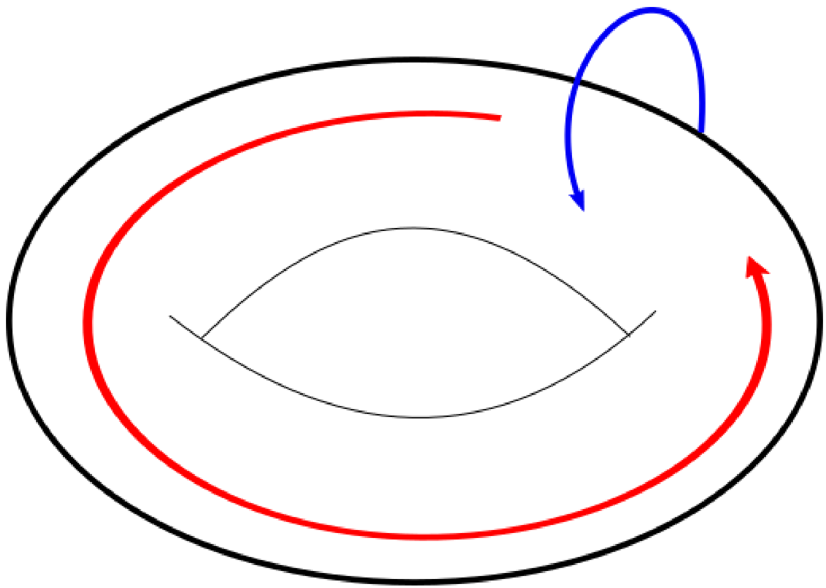
Any Möbius transformation is determined by where it sends 0, 1, and ∞.

In particular, only the identity map fixes three points.



Elliptic Curves

If X is an elliptic curve, then $\mathrm{Aut}X$ acts transitively on X and is therefore infinite. After picking a point $P \in X$ to be the group identity, we can regard X as the group $S^1 \times S^1$, which acts on itself automorphically by translation (as suggested by the arrows). So $S^1 \times S^1 \subseteq \mathrm{Aut}X$ hence $\#\mathrm{Aut}X = \infty$.



Curves of Genus ≥ 2

If X is a curve of genus $g = g(X) \geq 2$, then $\#\mathrm{Aut}(X) < \infty$. In fact, Hurwitz's Automorphisms Theorem says that $\#\mathrm{Aut}(X) \leq 84(g-1)$. We will proceed in two parts:

1. Proof of finiteness has two cases, depending on whether X is hyperelliptic or not.
2. Proof of upper bound follows by applying the Riemann-Hurwitz formula for the number of "ramification points" for a finite morphism $X \to C$ of curves.

Sketches of Proofs

Theorem: When X is a smooth projective curve with $g \geq 2$, $\#\text{Aut}X < \infty$.

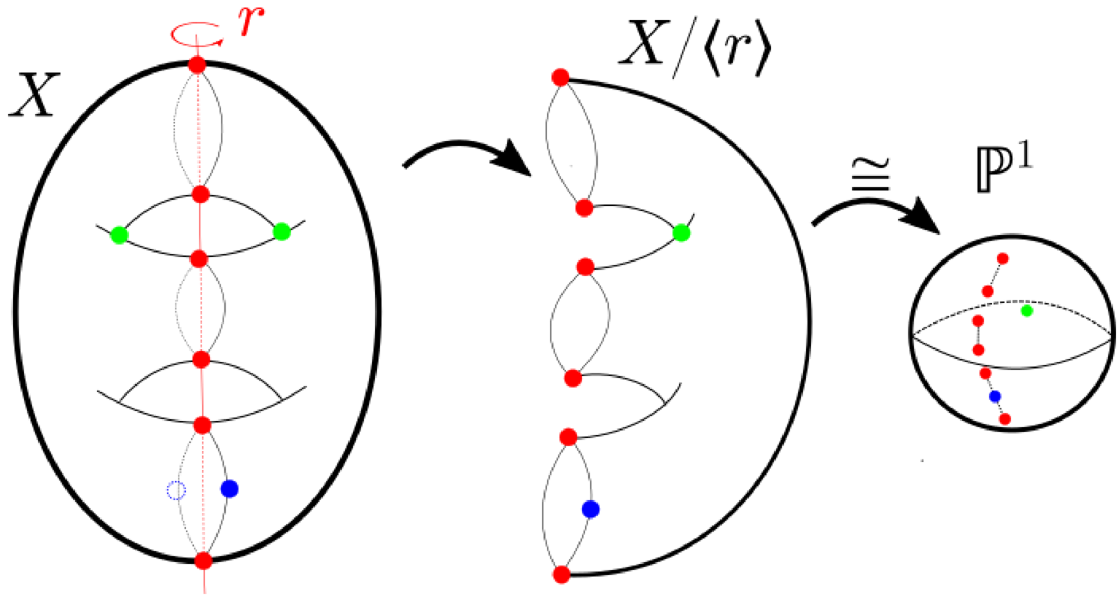
Hartshorne sketches 2 proofs:

1. One uses intersection theory on the surface $X \times X$; it is characteristic free and depends on:
   - Hodge index theorem;
   - Adjunction formula;
   - Construction of K_X in terms of $\Delta \subseteq X \times X$;
   - Finite generation of the Neron-Severi group.
2. The other assumes characteristic 0 and uses the fact that K_X defines a finite morphism $X \to \mathbb{P}^{g-1}$.

Miranda provides a proof assuming characteristic 0 and uses the finitely many (at most $g^3 - g$) Weierstrass points determined by K_X.

Hyperelliptic Curves

A curve X is *hyperelliptic* if there exists a unique degree-2 map from X to $\mathbb{P}^1$. A degree-2 map is two-to-one except for a finite set of points, called *ramification points*.



Finiteness of Aut$X$

1. If X is hyperelliptic, then Aut$X$ permutes the $2g+2$-many ramification points of the unique degree-2 map from X to $\mathbb{P}^1$. Only the hyperelliptic involution fixes them.
2. If X is not hyperelliptic, then Aut$X$ permutes the finitely many Weierstrass points determined by the canonical divisor on X. A *Weierstrass point* $P \in X$ is a point such that $\dim_{\mathbb{C}} \mathcal{L}(K_X - gP) > 0$. A non-hyperelliptic curve has more than $2g+2$ Weierstrass points, which is enough that only the identity map fixes them.

Upper bound on Aut$X$

Recall: $\#\mathrm{Aut}X \leq 84(g-1)$.

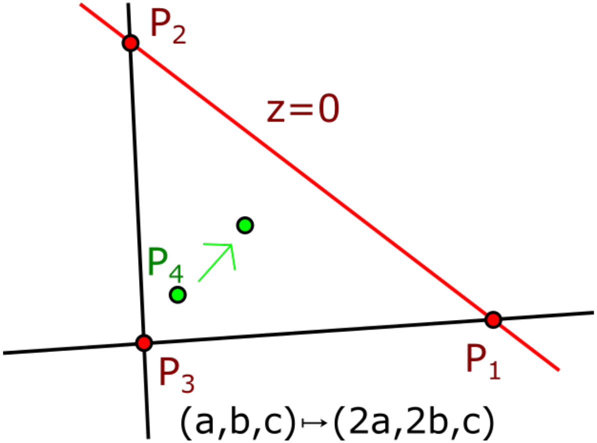
Proof of upper bound follows by applying the Riemann-Hurwitz formula to the quotient map $q : X \to (X/\mathrm{Aut}X)$. This states that

$$2g(X) - 2 = |\mathrm{Aut}X| \left(2g(X/\mathrm{Aut}X) - 2 + \sum_{i=1}^{s} \left(\frac{r_i - 1}{r_i} \right) \right)$$

where s is the number of ramification points of q and r_i is ramification index of each ramification point. The factor on the right achieves a minimum (positive) value of $\frac{1}{42}$ when $g(X/\mathrm{Aut}X) = 0$ and $-2 + \sum_{i=1}^{s} \left(\frac{r_i - 1}{r_i} \right) = \frac{1}{2} + \frac{2}{3} + \frac{6}{7} - 2 = \frac{1}{42}$.

Automorphisms of $\mathbb{P}^2_{\mathbb{C}}$

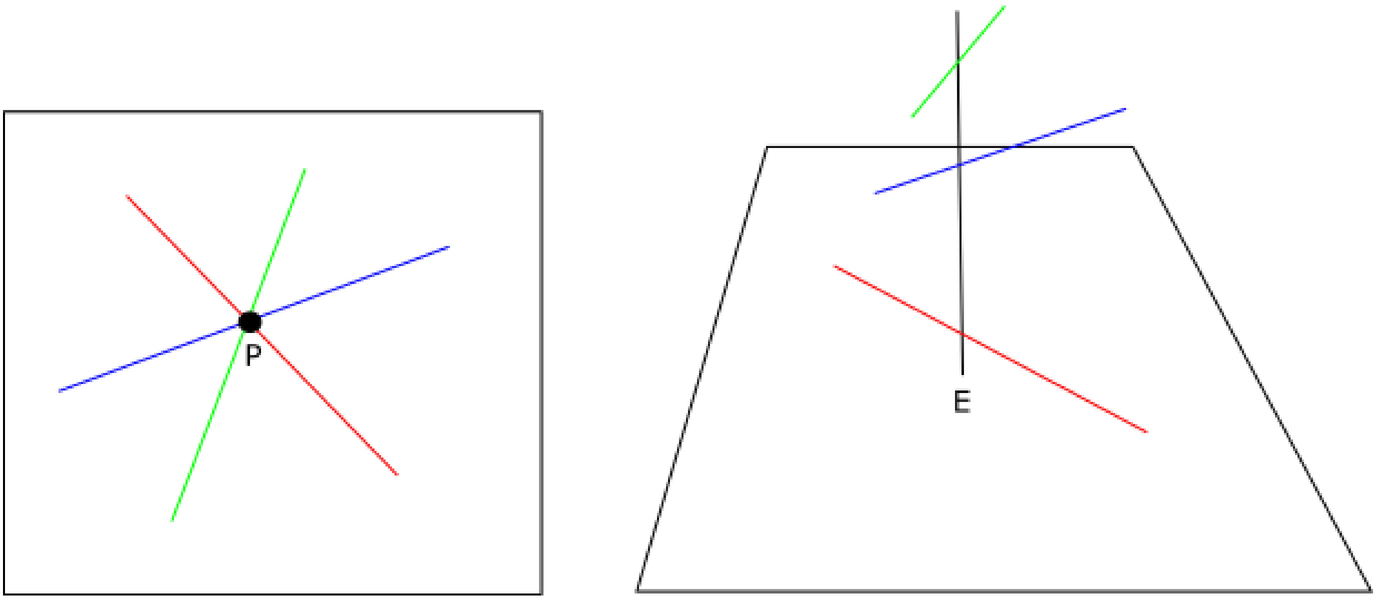
$\mathrm{Aut}(\mathbb{P}^2_{\mathbb{C}}) = \mathrm{PGL}(2) = \mathrm{GL}(3, \mathbb{C})/\mathbb{C}^* I_3$.
An automorphism of $\mathbb{P}^2$ can be determined by where it sends 4 *generic* points.



Example of an automorphism on $\mathbb{P}^2$ that fixes a point and a line.

Blowing Up Points

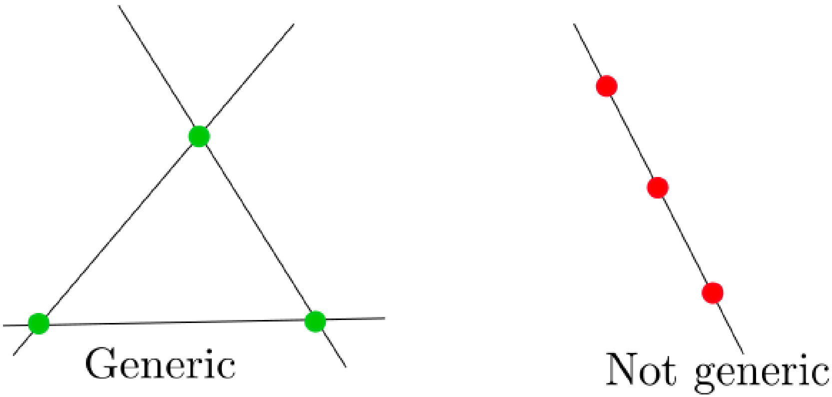
Question: What happens to the automorphism group of a plane, if we blow up some points?



$$\mathbb{P}^2 \setminus \{P\} \cong \widetilde{\mathbb{P}^2} \setminus E$$

Generic Position

- The answer depends on the relative position of the points. Let's assume the points are generic.



- Given points $P_1, \ldots, P_r$, we may assume up to change of coordinates that $P_i = (1, a_i, b_i)$. These points are generic if $\operatorname{tr.deg} \mathbb{Q}(a_1, b_1, \ldots, a_r, b_r) = 2r$.
- After change of coordinates, we may assume $P_1 = (1, 0, 0)$, $P_2 = (0, 1, 0)$, $P_3 = (0, 0, 1)$, and $P_4 = (1, 1, 1)$ with $\operatorname{tr.deg} \mathbb{Q}(a_5, b_5, \ldots, a_r, b_r) = 2(r - 5)$.

Automorphism Groups of Blown-Up Planes

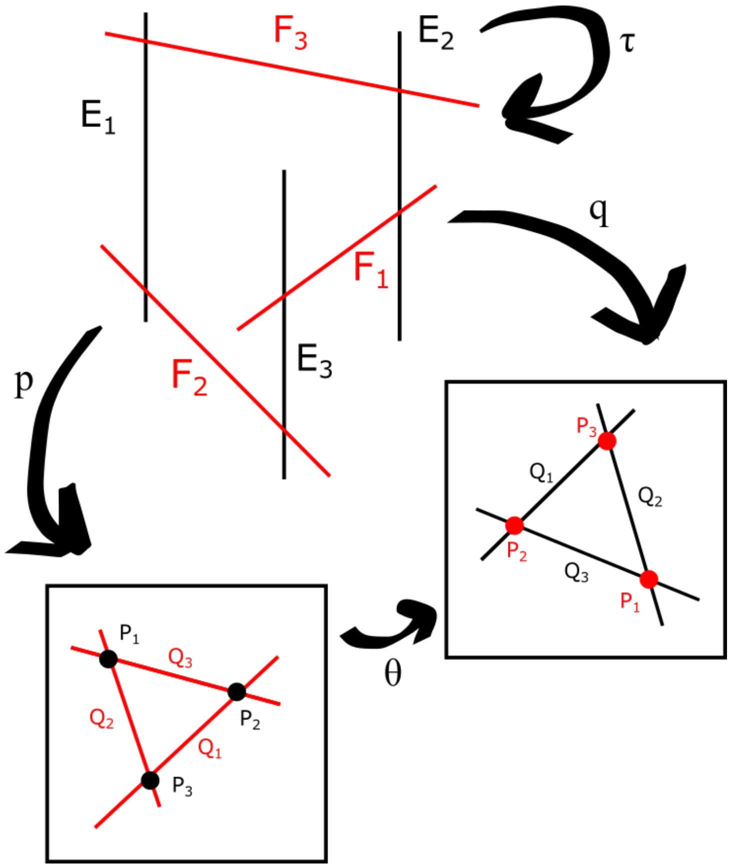
Let V_r be the plane $\mathbb{P}^2$ blown up at r generic points. Then we have the following table of automorphism groups by Koitabashi:

r	0	1	2
$\mathrm{Aut}V_r$	$\mathrm{PGL}(2)$	$\mathrm{PGL}(2)_{P_1}$	$\mathrm{PGL}(2)_{\{P_1,P_2\}}$

3	4	5
$\mathrm{PGL}(2)_{\{P_1,P_2,P_3\}} \rtimes (\mathbb{Z}/2\mathbb{Z})$	S_5	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 4}$

6	7	8	≥ 9
$\{\mathrm{id}\}$	$\mathbb{Z}/2\mathbb{Z}$ (Geiser involution)	$\mathbb{Z}/2\mathbb{Z}$ (Bertini involution)	$\{\mathrm{id}\}$

Note V_r is called a del Pezzo surface when $r \leq 8$.



Both p and q are blowdown maps from V_3 to $\mathbb{P}^2$.

The composition qp^{-1} defines a birational map θ on $U = \mathbb{P}^2 \setminus \{abc = 0\}$ given by $\theta(a,b,c) = \left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right)$.

This lifts to an automorphism $\tau \in \mathrm{Aut}V_3$ such that $p = q \circ \tau$.

Then $q = p \circ \tau$ and $\tau^2 = \mathrm{id}$ by [K].

So the following diagram commutes.

$$
\begin{array}{ccc}
V_3 & \xrightarrow{\ \tau\ } & V_3 \\
\downarrow p & \searrow q & \downarrow p \\
\mathbb{P}^2 & \overset{\theta}{\dashrightarrow} & \mathbb{P}^2 \\
\cup\!| & & \cup\!| \\
U & \xrightarrow{\ \theta\ } & U
\end{array}
$$

Let $H = \{\sigma \in \mathrm{Aut}V_3 : \{\sigma(E_1), \sigma(E_2), \sigma(E_3)\} = \{E_1, E_2, E_3\}\} \cong \mathrm{PGL}(2)_{\{P_1,P_2,P_3\}}$ Then $H \trianglelefteq \mathrm{Aut}V_3$ and $\mathrm{Aut}V_3 = H \rtimes \langle \tau \rangle$ where $\tau \circ \sigma \circ \tau = \sigma^{-1}$ for all $\sigma \in H$ by [K].

Intersection Product

Given two divisors D and D' on a smooth projective surface X, we denote the *intersection product* of D and D' as (D, D'). For two distinct reduced irreducible curves D and D', it is the sum of the intersection multiplicities at each point of $D \cap D'$.

The divisor class group is a quotient group of the free abelian group generated by prime divisors on X, modded out by linear equivalence. It is denoted $\mathrm{Pic} X$.

The *orthogonal group* of $\mathrm{Pic} X$ – denoted $O(\mathrm{Pic} X)$ – is the group of invertible linear transformations from $\mathrm{Pic} X$ to $\mathrm{Pic} X$ that preserve intersection product.

Weyl Groups

Let L be a generic line in $\mathbb{P}^2$. Lift L to V_r by $p^{-1}(L) =: E_0$, and define $R_0 = E_0 - E_1 - E_2 - E_3$, and $R_i = E_i - E_{i+1}$ for $1 \leq i \leq r-1$.

Then $(R_i, R_i) = -2$, $(R_i, K_r) = 0$ for all i, where $K_r = -3E_0 + E_1 + \cdots + E_r$ is the canonical divisor of V_r.

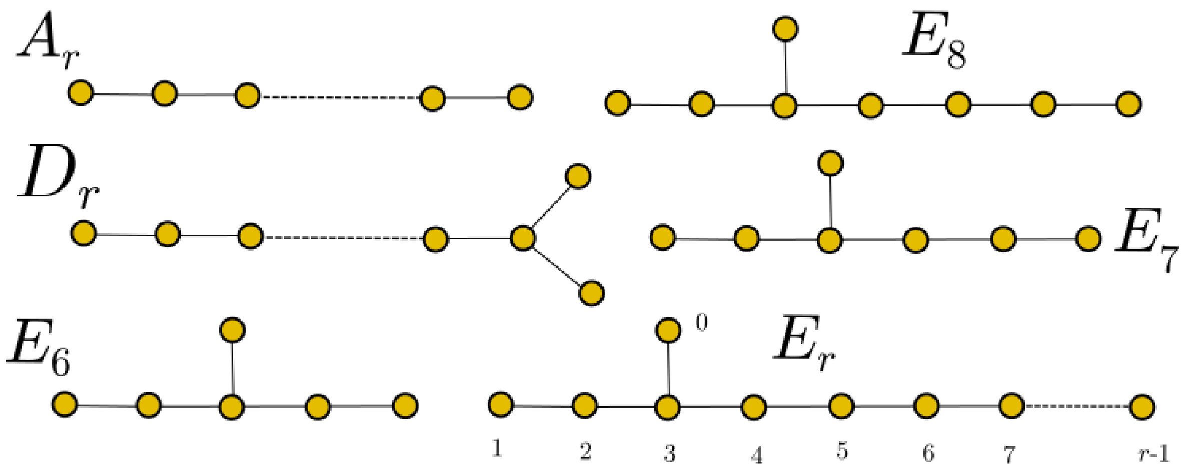
The Weyl group $W_r \leq O(\text{Pic}V_r)$ is generated by $s_i : \text{Pic}V_r \to \text{Pic}V_r$, where $s_i(x) = x + (x, R_i)R_i$. Note $s_i(R_i) = -R_i$ and $s_i(K_r) = K_r$ for all i.

The *root system* Φ_r on $\text{Pic}V_r \otimes \mathbb{R}$ is given by orbits of the R_i: $\Phi_r = \bigcup_{i=0}^{r-1} W_r \cdot R_i$. The R_i are the *simple roots* of Φ_r.

Weyl Groups Continued

The group homomorphism $\Psi_r : \text{Aut}V_r \to O(\text{Pic}V_r)$ is given by $\Psi_r(\sigma)(D) = (\sigma^{-1})^*(D)$ (the total transform of D under σ^{-1}) for all $D \in \text{Pic}V_r$. When $r \geq 4$, Ψ_r is injective and im $\Psi_r \trianglelefteq W_r$ [K].

The Weyl groups for $3 \leq r \leq 8$ are isomorphic to the finite Coxeter groups $A_1 \times A_2$, A_4, D_5, E_6, E_7, and E_8. The Weyl groups for $r \geq 9$ are infinite.



Dynamical Systems and Entropy

Entropy is used to understand the dynamics of an automorphism. A *dynamical system* consists of a manifold M (the *phase space*) and the powers of a smooth evolution function $f : M \to M$. Fatou and Julia were interested in the distinction between points of "regular behavior" and "wild" points.

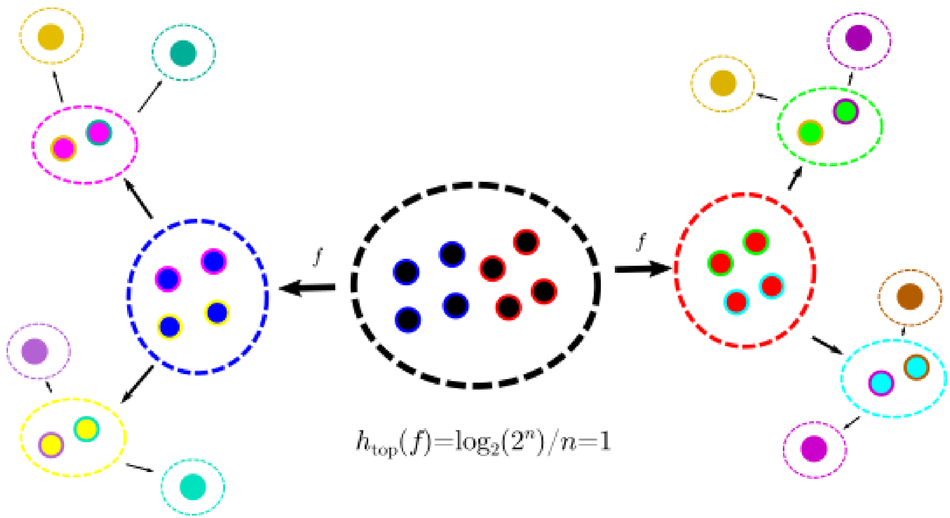
Let X be a compact metric space, and $f : X \to X$ a continuous map. Let $n \in \mathbb{Z}^+$ and let $\varepsilon > 0$. Then $x, y \in X$ have the same *orbit* of *period* n and *precision* ε if $\operatorname{dist}(f^j(x), f^j(y)) \leq \varepsilon$ for all $0 \leq j \leq n$.

If ε is fixed, then the number of orbits grows at most exponentially as n goes to ∞. The topological entropy $h_{\text{top}}(f)$ measures this growth as ε goes to 0.

Topological Entropy

Let $N(n, \varepsilon)$ be the maximal cardinality of a part E of X such that for each $y \in E$ there is an $x \in E$ in the same orbit as y of period n and precision ε. Then define $h_{\text{top}}(f, \varepsilon) = \limsup_{n \to \infty} \frac{1}{n} \log N(n, \varepsilon)$. This is usually $\log_2$ but this choice is arbitrary.

Then the *topological entropy* $h_{\text{top}}(f) := \lim_{\varepsilon \to 0} h_{\text{top}}(f, \varepsilon)$.



Entropy of Automorphisms

- An automorphism $f : S_r \to S_r$ induces a map on cohomology groups $f^* : H^2(S_r;\mathbb{Z}) \to H^2(S_r;\mathbb{Z}) = \bigoplus_{i=1}^r \mathbb{Z}[E_i]$. Note $H^2(S_r;\mathbb{Z}) \overset{\phi}{\cong} \mathrm{Pic} S_r$.
- $\lambda(f^*)$ denotes the spectral radius of f^*, which is the largest of the eigenvalues of f^*.
- By a theorem of Nagata, for every f^* there is a unique $w \in W_r$ such that $\phi \circ f^* = w \circ \phi$. We say w is *realized* by f.
- Gromov proved $h_{\text{top}}(f) = \log \lambda(f^*)$.

Coxeter Elements

The product of generators $(s_0, \ldots s_{r-1})$, taken one at a time in any order, is a *Coxeter element* $w \in W_r$. All Coxeter elements are conjugate and so have the same spectral radius. Thus it makes sense to talk about $\lambda_r := \lambda(w)$ as an inherent property of W_r.

Let $\mathbb{E}_r = \{h_{\text{top}}(f) : f : S_r \to S_r$ is a rational surface automorphism$\}$. Let $\Lambda_r = \{\lambda(w) : w \in W_r, r \geq 3\}$. Then by a theorem by Uehara [U], $\mathbb{E}_r = \log \Lambda_r$.

Positive Entropy

Theorem 1.1 in McMullen states that for $r \geq 10$, every Coxeter element $w \in W_r$ can be realized by a rational surface automorphism $F_r : S_r \to S_r$ with entropy $h_{\mathrm{top}}(F_r) = \log \lambda_r > 0$.
Furthermore, F_r can be chosen to have the following properties:

- The surface S_r is the blowup of r distinct points on a cuspidal cubic $C \subseteq \mathbb{P}^2$.
- There is a meromorphic nowhere-vanishing 2-form η on S_r with a simple pole on the proper transform C' of C.
- The homomorphism F_r^* satisfies $F_r^*(\eta) = \lambda_r \cdot \eta$.
- The Julia set $J^+(F_r)$ (the closure of the set of repelling periodic points) has measure zero, and every $z \in S_r \setminus J^+(F_r)$ converges under iteration to the unique singular point $p \in C'$.
- S_r equipped with the $\mathbb{Z}$-action generated by F_r is G-minimal in the sense of Manin.

Siegel Disks

A linear map $R : \mathbb{C}^2 \to \mathbb{C}^2$ is an *irrational rotation* if its eigenvalues lie on the unit circle and are multiplicatively independent.

Let $F : S \to S$ be a surface automorphism with a fixed point p. Then $U \subseteq S$ is a *Siegel disk* for F if $F(U) = U$ and $F|_U$ is analytically conjugate to an irrational rotation $R|_{(D^2)^2}$ where $D^2 \subseteq \mathbb{C}$ is the unit disk.

McMullen [M] constructed the first examples of automorphisms of projective algebraic varieties that have Siegel disks.

Siegel disks are related to linearizability of functions and Julia sets have important applications in complex dynamics– including applications regarding engineering, ecology, and physics.

References

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