The Geproci Property in Positive Characteristic

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What is Geproci?

A finite set Z in $\mathbb{P}^n_k$ is **geproci** if the projection $\overline{Z}$ of Z from a general point P to a hyperplane $H = \mathbb{P}^{n-1}_k$ is a complete intersection in H.

Geproci stands for **ge**neral **pro**jection is a **c**omplete **i**ntersection.

The only nontrivial examples known are for $n = 3$. In this case a hyperplane is a plane. A reduced set of points in a plane is a complete intersection if it is the transverse intersection of two algebraic curves, as in Figure 1.

For $\#Z = ab$ $(a \leq b)$, Z is (a, b)-geproci if $\overline{Z}$ is the intersection of a degree a curve and a degree b curve.

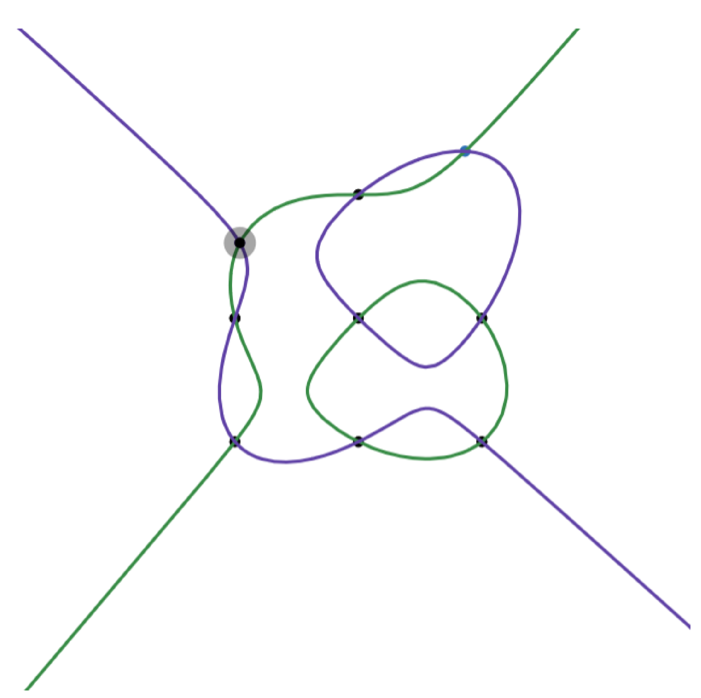

Figure 1. A (3,3) complete intersection

A **grid** is a geproci set whose image can be taken to be a complete intersection of two unions of lines. They are well known.

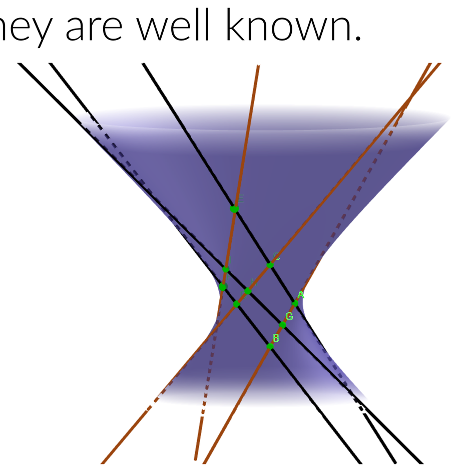

Figure 2. The 9 intersection points form a (3,3)-grid

The simplest nontrivial geproci sets are called half-grids.

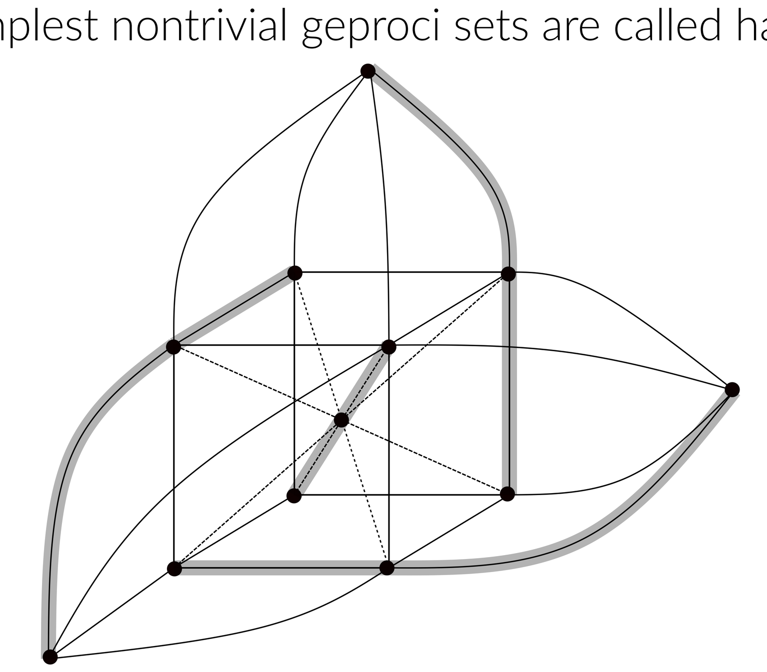

Figure 3. The D_4 configuration is an example of a (3,4)-geproci half-grid

The Unexpected Cone Property

The geproci property is closely linked to the unexpected cone property.

A 0-dimensional subscheme Z of $\mathbb{P}^n$ admits an **unexpected cone** of degree d if

$$\dim[I(Z) + I(P)^d]_d > \max\left\{0, \dim[I(Z)]_d - \binom{d+n-1}{n}\right\}$$

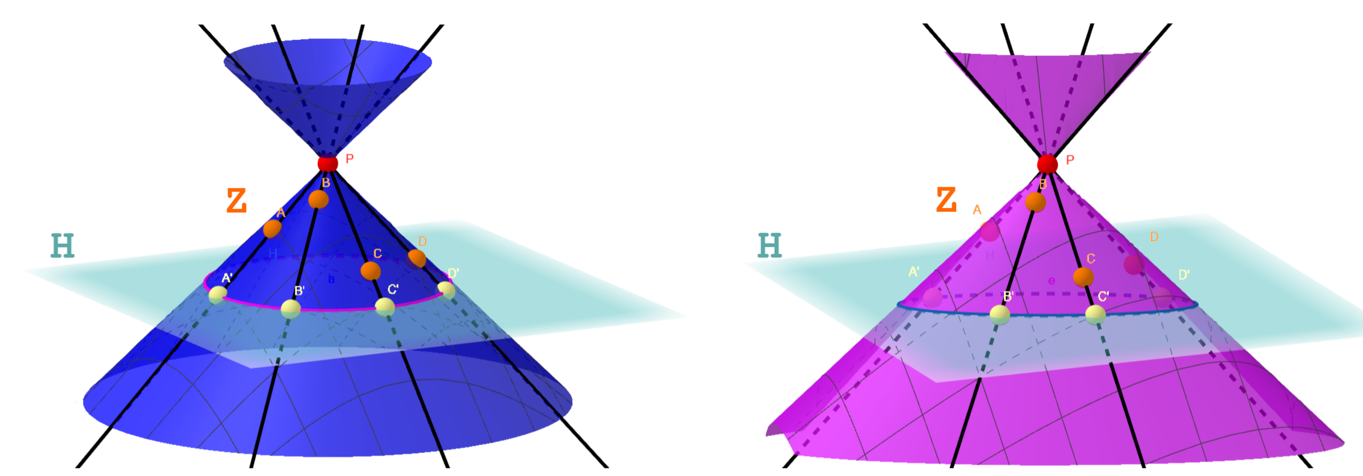

Figure 4. $\overline{Z}$ is a complete intersection of two conics, and Z lies on two cones.

The Positive Characteristic Setting

Theorem (K.) The set $Z = \mathbb{P}^3_{\mathbb{F}_q}$ is $(q+1, q^2+1)$-geproci in $\mathbb{P}^3_{\overline{\mathbb{F}}_q}$.

We can show this by constructing cones of degrees $q+1$ and q^2+1 containing Z and having a general vertex $P \in \mathbb{P}^3_{\overline{\mathbb{F}}_q}$. The cones must also have no points in common.

In fact these curves are unexpected:

$$1 = \dim[I(Z) + I(P)^{q+1}]_{q+1} > \max\left\{0, \dim[I(Z)]_{q+1} - \binom{q+3}{3}\right\} = 0$$

$$\dim[I(Z) + I(P)^{q^2+1}]_{q^2+1} > \max\left\{0, \dim[I(Z)]_{q^2+1} - \binom{q^2+3}{3}\right\} \qquad (q \geq 3)$$

$$7 = \dim[I(Z) + I(P)^{q^2+1}]_{q^2+1} > \max\left\{0, \dim[I(Z)]_{q^2+1} - \binom{q^2+3}{3}\right\} = 6 \qquad (q = 2)$$

The degree $q+1$ cone with vertex $P = (a, b, c, d) \in \mathbb{P}^3_{\overline{\mathbb{F}}_q}$ comes from the determinant of the matrix $\begin{pmatrix} a & b & c & d \\ a^q & b^q & c^q & d^q \\ x & y & z & w \\ x^q & y^q & z^q & w^q \end{pmatrix}$. The degree q^2+1 cone is special because it comes from **spreads**.

Spreads

A set of lines S in $\mathbb{P}^3$ **spread** if each point of $\mathbb{P}^3$ contained in exactly one line of S. Spreads are known to always exist over a finite field [1], and the Hopf fibration provides an example of a spread over $\mathbb{P}^3_{\mathbb{R}}$!

This makes $Z = \mathbb{P}^3_{\mathbb{F}_q}$ a geproci half grid. Note that when $q = 2$, we get a nontrivial $(3, 5)$-geproci set, which does not exist in characteristic 0.

PAMS Paper!

Partial Spreads

A **partial spread** with deficiency d is a set of $q^2 + 1 - d$ mutually skew lines.

A **maximal partial spread** is a partial spread of positive deficiency that is not contained in any larger partial spread.

Nontrivial Non-half-grids

Theorem (K.) The complement of a maximal partial spread with deficiency d is a $\{q+1, d\}$-geproci set. Furthermore, when $d > q+1$, this complement is a nontrivial non-half-grid.

For $q \geq 7$, there are maximal partial spreads of every deficiency in the interval $q - 1 \leq d \leq \frac{q^2+1}{2} - 6$ [3]. So this gives us a way of producing infinitely many nontrivial non-half-grids, which we do not have in characteristic 0 [2].

Nonreduced Schemes

The positive characteristic setting also gives us examples of geproci sets from nonreduced schemes. For example, let $A \in \mathbb{P}^3$ and $B \in Bl_A(\mathbb{P}^3)$ corresponding to the line L through A. Then

$$I(\{A, B\}) = I(L) + I(A)^2$$

and

$$I(\{\overline{A}, \overline{B}\}) = I(\overline{L}) + I(\overline{A})^2.$$

Example

Let char $k = 2$, and let

$X = \{(1:0:0:0)\times 2, (0:1:0:0)\times 2, (0:0:1:0)\times 2, (0:0:1:0)\times 2, (1:1:1:1)\}$,

where each infinitely-near point corresponds to the respective line through the given point and $(1:1:1:1)$. Then X is $(3, 3)$-geproci and is a nontrivial non-half-grid.

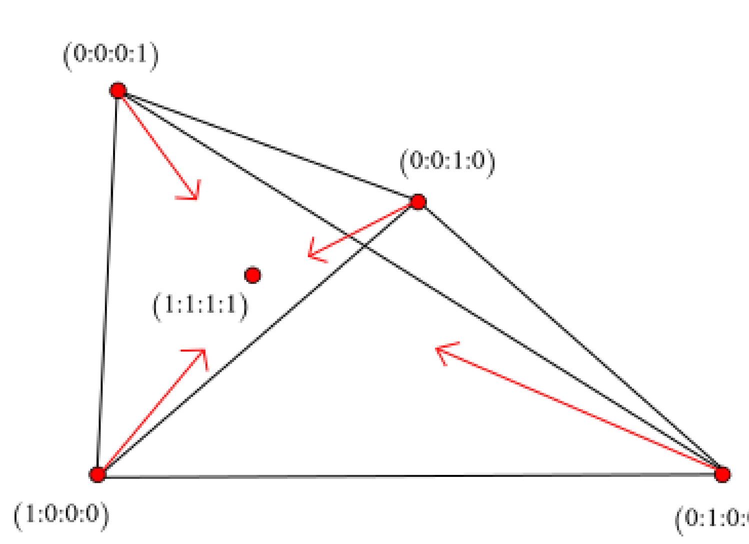

Figure 5. A $(3, 3)$-geproci nontrivial non-half-grid in characteristic 2

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