Oriented Steiner Systems

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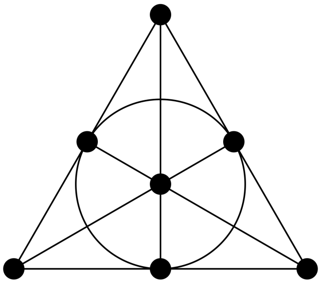
Steiner Triple Systems

Definition

A **Steiner triple system** is the pair $(\mathcal{S}, T)$ where $\mathcal{S}$ is a set of elements and $T \subset \mathcal{P}(\mathcal{S})$ is a set of triples, where for every pair of distinct elements $s, s' \in S$, there is a unique $t \in T$ such that $s, s' \in t$.

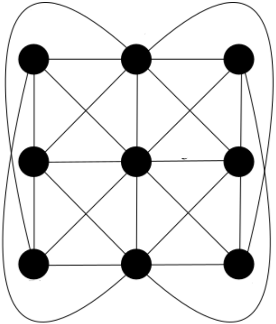


Size of an STS

If $\#\mathcal{S} = n$, then $\#T = n(n-1)/6$. Thus $n(n-1) \equiv 0 \mod 6$.
Furthermore, n must be odd because $\mathcal{S} \setminus \{s\}$ can be partitioned into pairs via $\{t \setminus \{s\} : s \in t \in T\}$.
Thus $n \equiv 1$ or $3 \mod 6$.

Theorem

A Steiner triple system of order n exists if and only if $n \equiv 1$ or $3 \mod 6$.



Quasigroups/Latin Squares

Definition

A **quasigroup** (aka a **Latin square**) is a nonempty set Q and a binary operation $\circ$ such that for any $x, y \in Q$, there exist unique $z, w \in Q$ such that $x \circ z = w \circ x = y$.

$\circ$	1	2	3
1	1	3	2
2	3	2	1
3	2	1	3

$\circ$	1	2	3	4
1	1	3	2	4
2	3	2	4	1
3	2	4	1	3
4	4	1	3	2

Two **commutative** quasigroups. The first is **idempotent**, the second is **half-idempotent**. There is a commutative idempotent quasigroup of any odd size, and a commutative half-idempotent quasigroup of any even size.

3 mod 6: The Bose Construction

Let $(Q, \circ) = \{1, \ldots, 2n+1\}$ be a commutative idempotent quasigroup. Then we can form an STS $(\mathcal{S}, T)$ where

$$\mathcal{S} = Q \times \mathbb{Z}/3\mathbb{Z}$$

and T comprises triples of the forms

$$\{(i,0),(i,1),(i,2)\} : i \in Q,$$
$$\{(i,a),(j,a),(i \circ j, a+1)\} : i \neq j \in Q, a \in \mathbb{Z}/3\mathbb{Z}.$$

1 mod 6: The Skolem Construction

Let $(Q, \circ) = \{1, \dots, 2n\}$ be a commutative half-idempotent quasigroup. Then we can form an STS $(\mathcal{S}, T)$ where

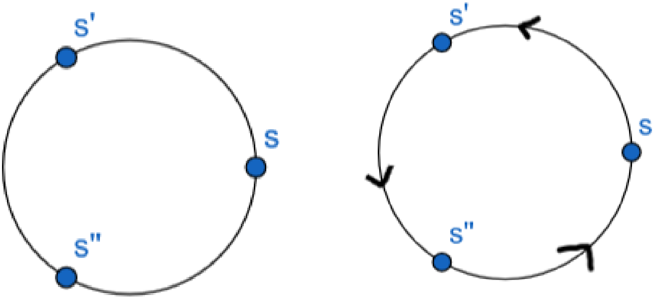
$$\mathcal{S} = (Q \times \mathbb{Z}/3\mathbb{Z}) \cup \{\infty\}$$

and T comprises triples of the forms

$$\{(i,0),(i,1),(i,2)\} : 1 \leq i \leq n,$$
$$\{(i,a),(j,a),(i \circ j, a+1)\} : i \neq j \in Q, a \in \mathbb{Z}/3\mathbb{Z},$$
$$\{\infty,(n+i,a),(i,a+1)\} : 1 \leq i \leq n, a \in \mathbb{Z}/3\mathbb{Z}.$$

Oriented Triples

We can create an orientation for a triple by drawing little arrows.



We can orient $(\mathcal{S}, T)$ by placing an orientation on every $t \in T$.

Oriented Steiner Triple Systems

Definition

Given a Steiner triple system $(\mathcal{S}, T)$, an **orientation on a triple** $t = \{x_1, x_2, x_3\} \in T$ is a function $f_t : t \times t \to \{-1, 0, 1\}$ such that

$$f_t(x_i, x_i) = 0,$$

$$f_t(x_i, x_j) = -f_t(x_j, x_i) \neq 0 \text{ when } i \neq j,$$

$$f_t(x_i, x_j) = -f_t(x_i, x_k) \neq 0 \text{ when } i, j, k \text{ are distinct.}$$

Furthermore, given an orientation f_t for all $t \in T$, an **orientation on the Steiner triple system** is a function $f : \mathcal{S} \times \mathcal{S} \to \{-1, 0, 1\}$ such that $f|_{t \times t} = f_t$ for all $t \in T$.

The Steiner Product

Now let $(\mathcal{S}, T, f)$ be an oriented Steiner triple system and consider the binary operation $\cdot \times \cdot : \mathcal{S} \to \mathbb{R}^{\mathcal{S}}$ such that for $s \neq s'$

$$s \times s' := f(s, s')s''$$

where $\{s, s', s''\}$ form a triple. If $s = s'$, then $s \times s' = 0$.

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Definition

Let $v, w \in \mathbb{R}^{\mathcal{S}}$. Then the **Steiner product** of v and w is

$$v \times w := \sum_{s \in \mathcal{S}} \sum_{s' \in \mathcal{S}} v_s w_{s'} (s \times s').$$

Connection to the Cross Product

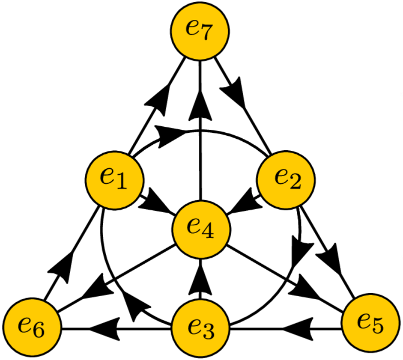
Definition

A **cross product** on the vector space $\mathbb{R}^n$ is a binary operation $\times$ that satisfies the following three properties:

1. $\times$ is bilinear,
2. $v \cdot (v \times w) = w \cdot (v \times w) = 0$ for all $v, w \in \mathbb{R}^n$,
3. $|v|^2 |w|^2 = |v \times w|^2 + (v \cdot w)^2$ for all $v, w \in \mathbb{R}^n$.

The cross product will only exist for $n = 3$ and $n = 7$.
The Steiner product will satisfy the first two properties, but will generally fail the third.

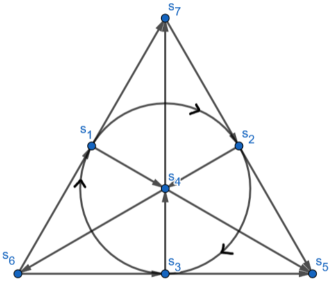
Example 1: The Octonions



$$(\mathcal{S}, T, f) = \{[1,2,3],[1,4,5],[1,7,6],[2,4,6],[2,5,7],[3,4,7],[3,6,5]\}$$

The imaginary part of the octonions form an oriented Steiner triple system. The Steiner product in this case is the cross product on $\mathbb{R}^7$.

Example 2: The Notonions



A different orientation on the same Steiner system yields different behavior.

$$(s_1 + s_5) \times (s_3 + s_7) = 0,$$

so

$$4 = |s_1 + s_5|^2 |s_3 + s_7|^2 \neq |(s_1 + s_5) \times (s_3 \times s_7)| + ((s_1 + s_5) \cdot (s_3 + s_7))^2 = 0.$$

Classifying Oriented Steiner Triple Systems

The STS's of sizes 7 and 9 are unique. There are 2 of size 13, 80 of size 15, and 11084874829 of size 19. How many when we have to keep track of orientation?

Definition

An **isomorphism** of oriented Steiner triple systems is a bijection $\phi : (\mathcal{S}, T, f) \to (\mathcal{S}', T', f')$ such that $\phi(t) \in T'$ for all $t \in T$ and $f(s_1, s_2) = f'(\phi(s_1), \phi(s_2))$ for all $s_1, s_2 \in \mathcal{S}$.

Classifying Oriented $STS(7)$'s and $STS(9)$'s

Via Maple computations, Peterson and I have arrived at the following results.

Theorem (K–, Peterson)

There are four isomorphism classes of oriented $STS(7)$'s; two classes have automorphism group $C_7 \rtimes C_3$, and two have automorphism group C_3.

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Theorem (K–, Peterson)

There are 16 isomorphism classes of oriented $STS(9)$'s; one class has automorphism group $C_3^2 \rtimes C_3$, one has automorphism group C_3^2, seven with C_3, and seven with trivial automorphism group.

Sizes 13 and above remain mysterious.

Aut($\mathcal{S}$, T, f) and Aut($\mathbb{R}^{\mathcal{S}}$, $\times$)

Every $\phi \in \mathrm{Aut}(\mathcal{S}, T, f)$ can be linearly extended to an automorphism $\phi^* \in \mathrm{Aut}(\mathbb{R}^{\mathcal{S}})$. Then

$$\phi^*(v \times w) = \phi^*(v) \times \phi^*(w),$$

so ϕ^* commutes with the Steiner product ($\phi^* \in \mathrm{Aut}(\mathbb{R}^{\mathcal{S}}, \times)$).

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Does every $f \in \mathrm{Aut}(\mathbb{R}^{\mathcal{S}}, \times)$ come from the linear extension of an automorphism of $(\mathcal{S}, T, f)$ this way?

The Steiner Product as Matrix Multiplication

Order the OSTS $(\mathcal{S}, T, f) = (\{s_1, \ldots, s_n\}, T, f)$. Denote

$$[v_1 s_1 + \cdots + v_n s_n] = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

and define the $n \times n$ matrix M by

$$M_{i,j} = s_i \times s_j.$$

Then

$$v \times w = \operatorname{tr}([v]^T M[w]).$$

A Rank Analysis

Given a $v \in \mathbb{R}^{\mathcal{S}}$, we can construct a matrix $A(v)$ such that the (i, j)-entry is the $\mathbb{R}$-coefficient of the s_j term of the Steiner product $v \times s_i$ for all $1 \leq i, j \leq n$.

Then $\ker A(v) = \{w \in \mathbb{R}^{\mathcal{S}} : v \times w = 0\}$. So v is a "zero-divisor" if and only if rank $A(v) < n - 1$.

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Example

Consider $v = s_1 + s_5$ from the notonion example. Then

$$A(v) = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

A Dynamics Problem

Given an OSTS $(\mathcal{S}, T, f)$ and $v, w_0 \in \mathbb{R}^{\mathcal{S}}$, defining $w_i := v \times w_{i-1}$ for $1 \leq i$, we get an ascending chain of vector subspaces

$$\langle v, w_0 \rangle \subseteq \langle v, w_0, w_1 \rangle \subseteq \langle v, w_0, w_1, w_2 \rangle \subseteq \cdots .$$

At what step does this chain achieve its maximum? Does that depend on the rank of $A(v)$?