New perspectives on geproci sets

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What is Geproci?

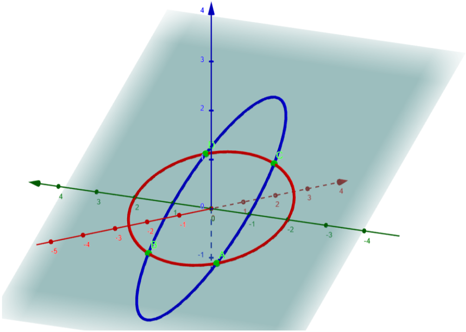
Definition

A finite set Z in $\mathbb{P}^n_k$ is **geproci** if the projection $\overline{Z}$ of Z from a general point P to a hyperplane $H = \mathbb{P}^{n-1}_k$ is a complete intersection in H.

Geproci stands for **ge**neral **pro**jection is a **c**omplete **i**ntersection.
The only nontrivial examples known are for $n = 3$. In this case a hyperplane is a plane. A reduced set of points in a plane is a complete intersection if it is the transverse intersection of two algebraic curves, like this.
For $\#Z = ab$ $(a \leq b)$, Z is (a, b)-geproci if $\overline{Z}$ is the intersection of a degree a curve and a degree b curve.

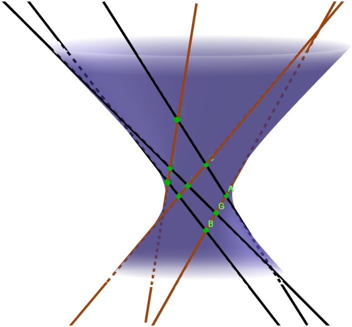
Trivial Case: Coplanar Points

A set of coplanar points can be geproci only if it is already a complete intersection in the plane it's on.



Trivial Cases: Grids

The easiest non-coplanar examples are grids, which are sets of points that form the intersection of two families of mutually-skew lines.

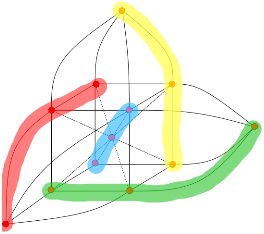


Summary of Nontrivial Cases

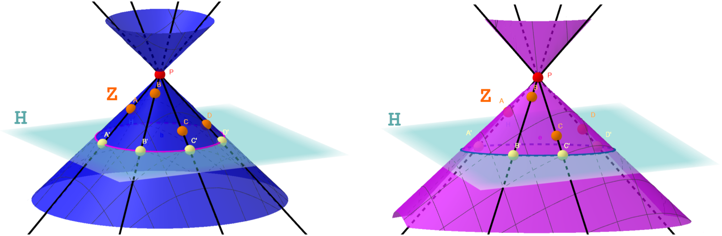
Half-Grids: A procedure is known for creating an (a, b)-geproci half-grid for $4 \leq a \leq b$, but it is not known what other examples there can be.

Non-Half-Grids: Before my thesis work, only a few examples were known and there was no known way to generate more.

Because of this, nontrivial non-half-grids have been mysterious; it's easier to get an idea of what a half-grid is like.



It is interesting when there is a cone through Z whose vertex is a general point P, and which meets H in a curve containing the projected image of Z. For Z to be (a, b)-geproci, there needs to be two such cones, of degrees a, b.



Geometry gets weird in positive characteristic p! For example, there's Fermat's Little Theorem and there's the Freshman's Dream (aka Frobenius): $(x + y)^p = x^p + y^p$. But this weirdness makes being geproci very natural!

Cones in $\mathbb{P}^3_{\mathbb{F}_q}$ of degree $a = q + 1$

Consider $Z = \mathbb{P}^3_{\mathbb{F}_q}$.

Note that $\#Z = \dfrac{q^4 - 1}{q - 1} = q^3 + q^2 + q + 1 = (q + 1)(q^2 + 1)$.

There is a unique degree $q + 1$ cone containing Z whose vertex is at a general point $P = (a, b, c, d) \in \mathbb{P}^3_k$, $k = \overline{\mathbb{F}}_q$. This cone meets every line through two points of $\mathbb{P}^3_{\mathbb{F}_q}$ transversely. It is given by

$$\begin{vmatrix} a & b & c & d \\ a^q & b^q & c^q & d^q \\ x & y & z & w \\ x^q & y^q & z^q & w^q \end{vmatrix} = 0$$

Is there a cone of degree $b = q^2 + 1$? There is!
Each line of $\mathbb{P}^3_{\mathbb{F}_q}$ contains $q + 1$ points. Can $\mathbb{P}^3_{\mathbb{F}_q}$ be partitioned by mutually-skew lines? Yes! Such a partition is called a **spread**, a name from combinatorics. The fibers S^1 of the Hopf fibration H map to the fibers $\mathbb{P}^1_{\mathbb{R}}$ of F, which give an example of a spread in $\mathbb{P}^3_{\mathbb{R}}$.

$$\begin{array}{ccc} S^3 & \xrightarrow{H} & S^2 \\ \downarrow A & & \downarrow = \\ \mathbb{P}^3_{\mathbb{R}} & \xrightarrow{F} & \mathbb{P}^1_{\mathbb{C}} \end{array}$$

For $\mathbb{P}^3_{\mathbb{F}_q}$, there are $q^2 + 1$ lines in the spread. The join of each line of the spread with P is our cone.

A Theorem

The following result gives a new method of constructing nontrivial geproci sets.

Theorem (K–)

The set of points $\mathbb{P}^3_{\mathbb{F}_q}$ is $(q+1, q^2+1)$-geproci in $\mathbb{P}^3_k$, where k is an algebraically closed field containing $\mathbb{F}_q$.

Note when $q = 2$, we get a non-trivial $(3, 5)$-geproci set! No nontrivial $(3, 5)$-geproci set exists in characteristic 0 [CFFHMSS], so this is new.

Partial Spreads

Definition

A **partial spread** of $\mathbb{P}^3_{\mathbb{F}_q}$ with **deficiency** d is a set of $q^2 + 1 - d$ mutually-skew lines. A **maximal partial spread** is a partial spread of positive deficiency that is not contained in any larger partial spread.

Maximal partial spreads give a way of producing infinitely many nontrivial non-half-grids.

Theorem (K–)

The complement of a maximal partial spread of deficiency d is a non-trivial $\{q+1, d\}$-geproci set. Furthermore, when $d > q+1$, the complement is a non-trivial non-half-grid.

In 1993 and 2002, Heden proved for $q \geq 7$ that there are maximal partial spreads of every deficiency d in the interval $q-1 \leq d \leq \frac{q^2+1}{2} - 6$.

The field $\mathbb{F}_7$ and Gorenstein Configurations

The maximal partial spreads in $\mathbb{P}^3_{\mathbb{F}_7}$ were classified by Soicher in 2000. They all comprise 45 lines, and their complements are configurations of 40 points.

Each complement is $(5, 8)$-geproci and is a non-half-grid. Furthermore, at least four of the fifteen are different up to projective equivalence and are Gorenstein! The four configurations I tested so far have stabilizers in $PGL(4, 7)$ of different sizes (10, 20, 60, and 120) and so are not projectively equivalent.

In characteristic 0, only one non-trivial Gorenstein configuration is known up to projective equivalence, also a configuration of 40 points [CFFHMSS].

Infinitely-Near Points

Definition

Let X be a smooth algebraic variety and let $P \in X$. The point Q is **infinitely-near** P if Q is on the exceptional locus of the blowup of X at P. (Intuitively, Q is a tangent direction at P.)

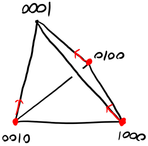
Abuse of notation: Technically, $Q \in \mathrm{BL}_P(X)$, but it is traditional to speak of infinitely-near points as if they were points of X itself.

Geproci With Infinitely-Near Points

Theorem (K–)

Let char $k = 2$. Let $Z = \{(1,0,0,0) \times 2, (0,1,0,0) \times 2, (0,0,1,0) \times 2\}$ (where $p_i \times 2$ represents an ordinary point $p_i \in \mathbb{P}^3_k$ and a point q_i infinitely near p_i), with the infinitely-near point at each ordinary point corresponding to the tangent along the line through p_i and $(0,0,0,1)$. Then Z is a $(2,3)$-geproci half-grid.

No $(2,3)$ half-grid is known in characteristic 0.

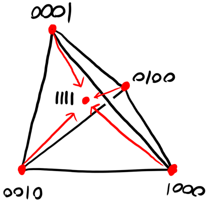


Another Example

Theorem (K–)

Let
$Z = \{(1,0,0,0)\times 2, (0,1,0,0)\times 2, (0,0,1,0)\times 2, (0,0,0,1)\times 2, (1,1,1,1)\}$, *with each infinitely-near point corresponding to the line containing* $(1,1,1,1)$. *Then* Z *is* $(3,3)$*-geproci. It is a non-trivial non-half-grid.*

No nontrivial $(3,3)$-geproci sets are known in characteristic 0.



Definition

A configuration of lines $\mathcal{L}$ in $\mathbb{P}^3$ is **dual-geproci** if the general projection of $\mathcal{L}$ into a plane H is dual to a complete intersection of points in H^*.

This projection function on lines can be thought of as a function within $Gr(2,4)$: Let $P \in \mathbb{P}^3$ and $H \in \mathbb{P}^{3*}$, then define

$$\pi_{P,H} : Gr(2,4) \setminus \Sigma_2(\mathcal{V}_P) \to \Sigma_{1,1}(\mathcal{V}_H)$$

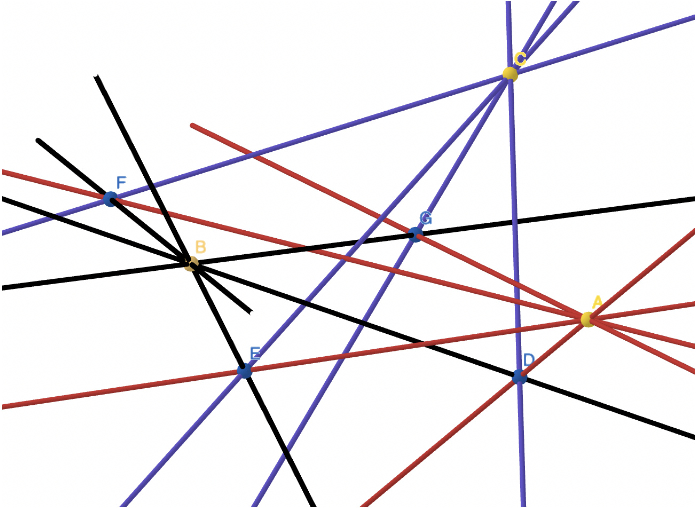
as

$$\pi_{P,H}(L) = \Sigma_{1,1}(\mathcal{V}_{\overline{PL}}) \cap \Sigma_{1,1}(\mathcal{V}_H),$$

where $\mathcal{V}_P$, $\mathcal{V}_{\overline{PL}}$, $\mathcal{V}_H$ are any flags containing P, $\overline{PL}$, and H, respectively. Then a finite set $\mathcal{L} \subseteq Gr(2,4)$ is dual-geproci if $\pi_{P,H}(\mathcal{L})$ is a complete intersection in the plane $\Sigma_{1,1}(\mathcal{V}_H)$ for a general $P \in \mathbb{P}^3$.

Complete Bipartite Graphs

So far, the only known examples of dual-geproci sets come from complete bipartite graphs in $\mathbb{P}^3$. These are the equivalent of grids, because the image of $\mathcal{L}$ is a complete intersection of two unions of lines in $\Sigma_{1,1}(\mathcal{V}_H)$.



Future Problems

1. Do infinitely-near points provide new examples of non-trivial geproci sets in characteristic 0?
2. Does taking higher-order infinitely-near points provide new examples of geproci sets?
3. Do maximal partial spreads provide new examples of geproci sets that work in characteristic 0?
4. Can geproci sets give new results on spreads?
5. Can we generalize known geproci configurations and known techniques to higher dimensions? For example $\#\mathbb{P}_{\mathbb{F}_q}^{2^t-1} = \prod_{i=0}^{t-1}(q^{2^i}+1)$.
6. What is a non-complete-bipartite example of of a dual-geproci set in $\mathbb{P}^3$?

Bibliography

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