Part 0: The Laplace Transform

Definition 0.1. Given a function $f(t)$ defined in the interval $[0, \infty)$, the Laplace transform of f is

$$\mathcal{L}\{f(t)\}(s) = \int_0^\infty f(t)e^{-st}\mathsf{d}t.$$

Example 0.1. Given $f(t) = 1$,

$$\mathcal{L}\{f(t)\}(s) = \int_0^\infty 1e^{-st}\mathsf{d}t = -\frac{1}{s}e^{-st}\bigg|_0^\infty = \frac{1}{s}.$$

Example 0.2. Given $f(t) = t$,

$$\mathcal{L}\{f(t)\}(s) = \int_0^\infty te^{-st}\mathsf{d}t.$$

Choosing

$$\begin{aligned} u &= t & \mathsf{d}v &= e^{-st}\mathsf{d}t \\ \mathsf{d}u &= \mathsf{d}t & v &= -\frac{1}{s}e^{-st} \end{aligned}$$

and using integration by parts $\int u\mathsf{d}v = uv - \int v\mathsf{d}u$, we get

$$-\frac{t}{s}e^{-st}\bigg|_0^\infty + \int_0^\infty \frac{1}{s}e^{-st}\mathsf{d}t = [0-0] - \frac{1}{s^2}e^{-st}\bigg|_0^\infty = \frac{1}{s^2}.$$

Example 0.3. Given $f(t) = e^{at}$,

$$\mathcal{L}\{f(t)\}(s) = \int_0^\infty e^{at}e^{-st}\mathsf{d}t = \int_0^\infty e^{(a-s)t}\mathsf{d}t = \frac{1}{a-s}e^{(a-s)t}\bigg|_0^\infty.$$

Note that $\lim_{t\to\infty} e^{(a-s)t} = \begin{cases} \infty & \text{if } a > s \\ 0 & \text{if } a < s \end{cases}$, so the Laplace transform can only be defined on the domain $a < s$, or (a, ∞).

Continuing in the case $a < s$,

$$\frac{1}{a-s}e^{(a-s)t}\bigg|_0^\infty = 0 - \frac{1}{a-s} = \frac{1}{s-a}.$$

Part 1: The Chart

Here is a table of common Laplace transforms:

$f(t)$	$\mathcal{L}\{f(t)\}(s)$
1	$\frac{1}{s}$
t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$\cos(kt)$	$\frac{s}{s^2+k^2}$
$\sin(kt)$	$\frac{k}{s^2+k^2}$
$\cosh(kt)$	$\frac{s}{s^2-k^2}$
$\sinh(kt)$	$\frac{k}{s^2-k^2}$

Note that $\cosh(t) = \dfrac{e^t + e^{-t}}{2}$ and $\sinh(t) = \dfrac{e^t - e^{-t}}{2}$.

Part 2: Linearity and Inverse Laplace Transforms

Also note that the Laplace transform acts *linearly* on its inputs: meaning that if we have functions $f(t)$ and $g(t)$ and constants a and b, then

$$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}.$$

Example 2.1.

$$\begin{aligned}\mathcal{L}\left\{3 + 5t^2 + t^7 - 8\cos(10t)\right\} = 3\mathcal{L}\{1\} = 5\mathcal{L}\{t^2\} + \mathcal{L}\{t^7\} - 8\mathcal{L}\{\cos(10t)\}\\ = \frac{3}{s} + \frac{10}{s^3} + \frac{7!}{s^8} - 8\frac{s}{s^2+100}.\end{aligned}$$

Example 2.2.

$$\mathcal{L}\left\{11 - 6\sin(\sqrt{3}t) + 15e^{-4t} + 15t^3\right\} = \frac{11}{s} - \frac{6\sqrt{3}}{s^2+3} + \frac{15}{s+4} + \frac{90}{s^4}.$$

The inverse Laplace transform also acts linearly: so given functions $F(s)$ and $G(s)$ and constants a and b,

$$\mathcal{L}^{-1}\{aF(s) + bG(s)\} = a\mathcal{L}^{-1}\{F(s)\} + b\mathcal{L}^{-1}\{G(s)\}.$$

To calculate inverse Laplace transforms, we will largely rely on The Chart in addition to rules like linearity instead of having an explicit formula.

$F(s)$	$\mathcal{L}^{-1}\{F(s)\}(t)$
$\frac{1}{s}$	1
$\frac{1}{s^n}$	$\frac{t^{n-1}}{(n-1)!}$
$\frac{1}{s-a}$	e^{at}
$\frac{s}{s^2+k^2}$	$\cos(kt)$
$\frac{1}{s^2+k^2}$	$\frac{\sin(kt)}{k}$
$\frac{s}{s^2-k^2}$	$\cosh(kt)$
$\frac{1}{s^2-k^2}$	$\frac{\sinh(kt)}{k}$

Example 2.3.

$$\begin{aligned}
&\mathcal{L}^{-1}\left\{\frac{1}{s^2+9} - \frac{10s}{s^2+11} + \frac{8}{s} - \frac{15}{s^5}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\right\} - 10\mathcal{L}^{-1}\left\{\frac{s}{s^2+11}\right\} + 8\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - 15\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} \\
&= \frac{\sin(3t)}{3} - 10\cos(\sqrt{11}t) + 8 - 15t.
\end{aligned}$$

Example 2.4.

$$\begin{aligned}
&\mathcal{L}^{-1}\left\{\frac{5}{s^5} + \frac{1}{6s^6} + \frac{7s}{4s^2+100} + \frac{15}{2s-100}\right\} \\
&= 5\mathcal{L}^{-1}\left\{\frac{1}{s^5}\right\} + \frac{1}{6}\mathcal{L}^{-1}\left\{\frac{1}{s^6}\right\} + \frac{7}{4}\mathcal{L}^{-1}\left\{\frac{s}{s^2+25}\right\} + \frac{15}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-50}\right\} \\
&= 5\frac{t^4}{4!} + \frac{1}{6}\cdot\frac{t^5}{5!} + \frac{7}{4}\cos(5t) + \frac{15}{2}e^{50t}.
\end{aligned}$$

Part 3: The First Translation Theorem

Theorem 3.1. Given $\mathcal{L}\{f(t)\}(s) = F(s)$ and a is a contstant, then

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a).$$

To put this another way,

$$\mathcal{L}^{-1}\{F(s-a)\} = e^{at}f(t).$$

Example 3.1. If we take $\mathcal{L}\{e^{5t}t^3\}$ and ignore the e^{5t}, we have $\mathcal{L}\{t^3\}$, which is $\dfrac{3!}{s^4}$. Now using that e^{5t} to shift the result by 5, we get $\dfrac{3!}{(s-5)^4}$ as our answer.

Example 3.2. If we look at $\mathcal{L}\{e^{-2t}\cos(7t)\}$ and ignore the e^{-2t}, we are left with $\mathcal{L}\{\cos(7t)\}$, which is $\dfrac{s}{s^2+49}$. Now using the e^{-2t} to shift the result by -2, we get $\dfrac{s+2}{(s+2)^2+49}$ as our answer.

Example 3.3. If we look at $\mathcal{L}^{-1}\left\{\dfrac{9}{(s+10)^2+4}\right\}$, we can first ignore the shift by 10 and look at $\mathcal{L}^{-1}\left\{\dfrac{9}{s^2+4}\right\}$. From the chart, we can see that this is $\dfrac{9}{2}\sin(2t)$. Now to compensate for the shift by 10, we need to multiply our result by e^{-10t} to get $\dfrac{9}{2}e^{-10t}\sin(2t)$.

Example 3.4. Now let's look at $\mathcal{L}^{-1}\left\{\dfrac{9s}{(s+10)^2+4}\right\}$. Now we can't just ignore the shift by 10 because the numerator is just $9s$, which does not have the shift of 10. We need to rewrite the numerator in terms of $s+10$. That is, we need to find a number A such that $9s = 9(s+10) + A$. Solving for A yields $A = -90$, so we must alter our expression to

$$\mathcal{L}^{-1}\left\{\frac{9(s+10)-90}{(s+10)^2+4}\right\} = \mathcal{L}^{-1}\left\{\frac{9(s+10)}{(s+10)^2+4}\right\} - \mathcal{L}^{-1}\left\{\frac{90}{(s+10)^2+4}\right\}.$$

Now we can ignore the shift by 10 in the function and solve for

$$\mathcal{L}^{-1}\left\{\frac{9s}{s^2+4}\right\} - \mathcal{L}^{-1}\left\{\frac{90}{s^2+4}\right\} = 9\cos(2t) - \frac{90}{2}\sin(2t).$$

Finally, we need to adjust our result to compensate for the shift by 10: we do this by multiplying our result by e^{-10t}. So we will end up getting $e^{-10t}(9\cos(2t) - 45\sin(2t))$.

Part 4: Partial Fractions and Completing the Square

Given polynomials $f(s)$ and $g(s)$, we can rewrite the rational function $\dfrac{f(s)}{g(s)}$ as a sum of simpler fractions based on the factors of the denominator $g(s)$. This will be useful in computing inverse Laplace transforms.

Example 4.1. Given $F(s) = \dfrac{2s+5}{s^2-1}$, we can see the denominator factors as $(s+1)(s-1)$. So we can begin rewriting F as

$$\frac{2s+5}{s^2-1} = \frac{A}{s+1} + \frac{B}{s-1},$$

and so

$$\frac{2s+5}{s^2-1} = \frac{A(s-1)}{(s+1)(s-1)} + \frac{B(s+1)}{(s-1)(s+1)} = \frac{A(s-1)+B(s+1)}{s^2-1}$$

and thus

$$2s+5 = A(s-1) + B(s+1).$$

There are two ways of solving for A and B. One way is to get a system of equations:

$$2s+5 = As - A + Bs + B = (A+B)s + (-A+B)$$

and so

$$\begin{aligned} 2 &= A + B \\ 5 &= -A + B. \end{aligned}$$

Solving this system of equations gives $A = -\frac{3}{2}$ and $B = \frac{7}{2}$.

Another way to find A and B is to plug in values for s in the equation $2s+5 = A(s-1) + B(s+1)$ that cause either the A term or B term to disappear: when $s = 1$, we have $7 = 2B$ and so $B = \frac{7}{2}$, and when $s = -1$ we get $3 = A(-2)$ and so $A = -\frac{3}{2}$.

So we can rewrite the function as follows:

$$\frac{2s+5}{s^2-1} = -\frac{3}{2(s+1)} + \frac{7}{2(s-1)}.$$

We can use this to calculate $\mathcal{L}^{-1}\{F(s)\}$.

$$\mathcal{L}^{-1}\left\{\frac{2s+5}{s^2-1}\right\} = -\frac{3}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + \frac{7}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} = -\frac{3}{2}e^{-t} + \frac{7}{2}e^{t}.$$

Note: for this one you can also use

$$\mathcal{L}^{-1}\left\{\frac{2s+5}{s^2-1}\right\} = 2\mathcal{L}^{-1}\left\{\frac{s}{s^2-1}\right\} + 5\mathcal{L}^{-1}\left\{\frac{1}{s^2-1}\right\} = 2\cosh(t) + 5\sinh(t).$$

Example 4.2. Given $F(s) = \dfrac{s+1}{s^4+5s^3+6s^2}$, we can factor the denominator as $s^2(s^2+5s+6) = s^2(s+2)(s+3)$, so we can set up the partial fraction decomposition

$$\frac{s+1}{s^4+5s^3+6s^2} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+2} + \frac{D}{s+3}.$$

We then get

$$s+1 = As(s+2)(s+3) + B(s+2)(s+3) + Cs^2(s+3) + Ds^2(s+2).$$

Plugging in $s=0$, we get $1 = 6B$ and so $B = \frac{1}{6}$. Plugging in $s=-2$, we get $-1 = 4C$, so $C = -\frac{1}{4}$. Plugging in $s=-3$, we get $-2 = -9D$, so $D = \frac{2}{9}$.

To solve for A, we must expand the equation to

$$s+1 = As^3 + 5As^2 + 6As + \frac{1}{6}s^2 + \frac{5}{6}s + 1 - \frac{1}{4}s^3 - \frac{3}{4}s^2 + \frac{2}{9}s^3 + \frac{4}{9}s^2.$$

By grouping up all the s^3 terms on both sides, we get

$$0s^3 = As^3 - \frac{1}{4}s^3 + \frac{2}{9}s^3,$$

and so $0 = A - \frac{1}{4} + \frac{2}{9}$, so $A = \frac{1}{36}$. We can also get this by grouping up the s^2 and s terms instead: we would get $0s^2 = 5As^2 + \frac{1}{6}s^2 - \frac{3}{4}s^2 + \frac{4}{9}s^2$ and $s = 6As + \frac{5}{6}s$, respectively.

So we can finish the partial fraction decomposition as

$$\frac{s+1}{s^4+5s^3+6s^2} = \frac{1}{36s} + \frac{1}{6s^2} - \frac{1}{4(s+2)} + \frac{2}{9(s+3)}.$$

We can use this to calculate $\mathcal{L}^{-1}\{F(s)\}$.

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{s+1}{s^4+5s^3+6s^2}\right\} &= \frac{1}{36}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \frac{1}{6}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} + \frac{2}{9}\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} \\ &= \frac{1}{36} + \frac{1}{6}t - \frac{1}{4}e^{-2t} + \frac{2}{9}e^{-3t}.\end{aligned}$$

We can also use the method of completing the square in the case there's an irreducible polynomial in the denominator. Recall that the completing the square method rewrites the polynomial s^2+bs+c as $\left(s+\frac{b}{2}\right)^2 + c - \frac{b^2}{4}$.

Example 4.3. Given $F(s) = \frac{3s+1}{s^2-10s+34}$, notice that the denominator is irreducible over the reals. So in order to compute $\mathcal{L}^{-1}\left\{\frac{3s+1}{s^2-10s+34}\right\}$, we will need to complete the square:

$$s^2 - 10s + 34 = (s-5)^2 + 34 - 25 = (s-5)^2 + 9.$$

Now we have $\mathcal{L}^{-1}\left\{\frac{3s+1}{(s-5)^2+9}\right\}$, which will require the First Translation Theorem. First we need to rewrite the numerator in terms of $s-5$: $3s+1 = 3(s-5)+16$. Now we can solve

$$\mathcal{L}^{-1}\left\{\frac{3(s-5)+16}{(s-5)^2+9}\right\} = \mathcal{L}^{-1}\left\{\frac{3(s-5)}{(s-5)^2+9}\right\} + \mathcal{L}^{-1}\left\{\frac{16}{(s-5)^2+9}\right\}$$

by first ignoring the shift by 5 and looking at

$$\mathcal{L}^{-1}\left\{\frac{3s}{s^2+9}\right\}+\mathcal{L}^{-1}\left\{\frac{16}{s^2+9}\right\}=3\cos(3t)+\frac{16}{3}\sin(3t).$$

The First Translation Theorem says to compensate for the shift by multiplying this result by e^{5t}, so our answer is $e^{5t}\left(3\cos(3t)+\frac{16}{3}\sin(3t)\right)$.

Example 4.4. Given $F(s)=\frac{s^2+1}{s^4+6s^3+13s^2}$, the denominator factors as $s^2(s^2+6s+13)$, so we can set up the partial fraction decomposition

$$\frac{s^2+1}{s^2(s^2+6s+13)}=\frac{A}{s}+\frac{B}{s^2}+\frac{Cs+D}{s^2+6s+13}.$$

Once we get all the fractions over a common denominator, we will get the equation

$$s^2+1=As(s^2+6s+13)+B(s^2+6s+13)+(Cs+D)s^2.$$

Plugging in $s=0$, we get

$$1=13B,$$

so $B=\frac{1}{13}$. Since $s^2+6s+13$ has no real roots, let's set up a system of equations by expanding

$$s^2+1=As^3+6As^2+13As+Bs^2+6Bs+13B+Cs^3+Ds^2$$

and grouping by powers of s:

$$\begin{aligned}
s^3 &: 0=A+C\\
s^2 &: 1=6A+B+D\\
s^1 &: 0=13A+6B\\
s^0 &: 1=13B.
\end{aligned}$$

Knowing $B=\frac{1}{13}$, the third equation gives us $13A+\frac{6}{13}=0$, so $A=-\frac{6}{169}$. Then the first equation becomes $0=-\frac{6}{169}+C$, so $C=\frac{6}{169}$. Finally, the second equation becomes $1=\frac{36}{169}+\frac{1}{13}+D$, so $D=\frac{120}{169}$.

So we can finish the partial fraction decomposition as

$$\frac{s^2+1}{s^2(s^2+6s+13)}=-\frac{6}{169s}+\frac{1}{13s^2}+\frac{\frac{6}{169}s+\frac{120}{169}}{s^2+6s+13}.$$

To avoid excessive fractions, let's just write this as

$$\frac{1}{169}\left(-\frac{6}{s}+\frac{13}{s^2}+\frac{6s+120}{s^2+6s+13}\right).$$

The inverse Laplace transforms of the $-\frac{6}{s}$ and $\frac{13}{s^2}$ are relatively straightforward: they are -6 and $13t$, respectively. The $\frac{6s+120}{s^2+6s+13}$ will require completing the square:

$$s^2 + 6s + 13 = (s+3)^2 + 13 - 9 = (s+3)^2 + 4.$$

So we get

$$\frac{6s+120}{s^2+6s+13} = \frac{6s+120}{(s+3)^2+4} = \frac{6(s+3)+102}{(s+3)^2+4}.$$

Ignoring the shift by 3, we get

$$\frac{6s+102}{s^2+4} = \frac{6s}{s^2+4} + \frac{102}{s^2+4},$$

and

$$\mathcal{L}^{-1}\left\{\frac{6s}{s^2+4} + \frac{102}{s^2+4}\right\} = 6\cos(2t) + \frac{102}{2}\sin(2t).$$

The First Translation Theorem says to adjust for the shift by 3 by multiplying the result by e^{-3t}, so

$$\mathcal{L}^{-1}\left\{\frac{6s+120}{s^2+6s+13}\right\} = e^{-3t}(6\cos(2t) + 51\sin(2t)).$$

Putting it all together, we get

$$\mathcal{L}^{-1}\left\{\frac{s^2+1}{s^4+6s^3+13s^2}\right\} = \frac{1}{169}(-6 + 13t + 6e^{-3t}\cos(2t) + 51e^{-3t}\sin(2t)).$$

Part 5: The Derivative Theorem and Solving IVPs

Theorem 5.1. Let $\mathcal{L}\{y(t)\}(s) = Y(s)$. Then

$$\mathcal{L}\{y^{(n)}(t)\}(s) = s^nY(s) - s^{n-1}y(0) - s^{n-2}y'(0) - s^{n-3}y''(0) - \cdots - sy^{(n-2)}(0) - y^{(n-1)}(0).$$

In particular,

$$\begin{aligned}
\mathcal{L}\{y'(t)\}(s) &= sY(s) - y(0) \\
\mathcal{L}\{y''(t)\}(s) &= s^2Y(s) - sy(0) - y'(0) \\
\mathcal{L}\{y'''(t)\}(s) &= s^3Y(s) - s^2y(0) - sy'(0) - y''(0) \\
\mathcal{L}\{y^{(4)}(t)\}(s) &= s^4Y(s) - s^3y(0) - s^2y'(s) - sy''(0) - y'''(0) \\
&\vdots
\end{aligned}$$

and so forth.

This lets us solve IVPs.

Example 5.1. Consider $y' - y = 1$ with the condition $y(0) = 0$. Applying $\mathcal{L}$ to both sides give us:

$$\begin{aligned} \mathcal{L}\{y' - y\} &= \mathcal{L}\{1\} \\ \mathcal{L}\{y'\} - \mathcal{L}\{y\} &= \frac{1}{s} \\ sY(s) - y(0) - Y(s) &= \frac{1}{s}. \end{aligned}$$

Since $y(0) = 0$, our equation is $sY(s) - Y(s) = \frac{1}{s}$. Then we can solve for $Y(s)$.

$$\begin{aligned} Y(s)(s-1) &= \frac{1}{s} \\ Y(s) = \frac{1}{(s-1)s} &= \frac{1}{s-1} - \frac{1}{s}. \end{aligned}$$

Then by taking inverse Laplace transforms, we get

$$y = \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = e^t - 1.$$

Example 5.2. Consider $y'' + 5y' + 4y = 0$ such that $y(0) = 1$ and $y'(0) = 0$. Start by taking the Laplace transform of both sides:

$$\begin{aligned} \mathcal{L}\{y'' + 5y' + 4y\} &= \mathcal{L}\{0\} \\ \mathcal{L}\{y''\} + 5\mathcal{L}\{y'\} + 4\mathcal{L}\{y\} &= 0 \\ s^2Y(s) - sy(0) - y'(0) + 5(sY(s) - y(0)) + 4Y(s) &= 0. \end{aligned}$$

Plugging in 1 for $y(0)$ and 0 for $y'(0)$ gives us

$$s^2Y(s) - s + 5sY(s) - 5 + 4Y(s) = 0.$$

Solving for $Y(s)$ gives us

$$Y(s) = \frac{s+5}{s^2+5s+4} = \frac{s+5}{(s+4)(s+1)} = -\frac{1}{3(s+4)} + \frac{4}{3(s+1)}.$$

Then

$$y = -\frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\} + \frac{4}{3}\mathcal{L}\left\{\frac{1}{s+1}\right\} = -\frac{e^{-4t}}{3} + \frac{4e^{-t}}{3}.$$

Example 5.3. Consider $y'' + 9y = e^t$ such that $y(0) = 0$ and $y'(0) = 0$. Taking the Laplace transform on both sides gives us

$$\begin{aligned} \mathcal{L}\{y'' + 9y\} &= \mathcal{L}\{e^t\} \\ \mathcal{L}\{y''\} + 9\mathcal{L}\{y\} &= \frac{1}{s-1} \\ s^2Y(s) - sy(0) - y'(0) + 9Y(s) &= \frac{1}{s-1} \\ s^2Y(s) + 9Y(s) &= \frac{1}{s-1} \end{aligned}$$

$$Y(s) = \frac{1}{(s-1)(s^2+9)} = \frac{1}{10(s-1)} - \frac{s+1}{10(s^2+9)}.$$

Taking the inverse Laplace transform gives us

$$y = \frac{1}{10}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - \frac{1}{10}\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\} - \frac{1}{10}\mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\right\} = \frac{e^t}{10} - \frac{\cos(3t)}{10} - \frac{\sin(3t)}{30}.$$

Example 5.4. Consider $y'' + 4y' + 5y = 0$ with the initial conditions $y(0) = 2$ and $y'(0) = 3$. Applying $\mathcal{L}$ on both sides yields

$$s^2Y(s) - sy(0) - y'(0) + 4sY(s) - 4y(0) + 5Y(s) = 0.$$

Plugging in 2 for $y(0)$ and 3 for $y'(0)$ gives us

$$s^2Y(s) - 2s - 3 + 4sY(s) - 12 + 5Y(s) = 0.$$

Solving for $Y(s)$ yields

$$Y(s) = \frac{2s+15}{s^2+4s+5} = \frac{2s+15}{(s+2)^2+1} = \frac{2(s+2)+11}{(s+2)^2+1} = \frac{2(s+2)}{(s+2)^2+1} + \frac{11}{(s+2)^2+1}.$$

By ignoring the shift by 2 and taking the inverse Laplace transform, we get

$$\mathcal{L}^{-1}\left\{\frac{2s}{s^2+1}\right\} + \mathcal{L}^{-1}\left\{\frac{11}{s^2+1}\right\} = 2\cos(t) + 11\sin(t).$$

Applying the First Translation Theorem, we multiply this result by e^{-2t} to get

$$y = e^{-2t}(2\cos(t) + 11\sin(t))$$

Part 6: Step Functions

Definition 6.1. The unit step function is

$$\mathcal{U}(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}.$$

$\mathcal{U}$ is "off"/"asleep" when t is negative and "turns on"/"wakes up" as soon as t hits 0. Usually we will work with the translated step function $\mathcal{U}(t-a)$, which wakes up at $t = a$.

The step function provides an alternate way to express piecewise functions. For instance

$$f(t) = \begin{cases} g(t) & \text{if } 0 \leq t < a \\ h(t) & \text{if } t \geq a \end{cases}$$

is

$$g(t)(1 - \mathcal{U}(t-a)) + h(t)\mathcal{U}(t-a).$$

Let's examine why: when $0 \leq t < a$, $\mathcal{U}(t-a) = 0$, so

$$g(t)(1 - \mathcal{U}(t-a)) + h(t)\mathcal{U}(t-a) = g(t)(1-0) + h(t)(0) = g(t),$$

which matches $f(t)$. And when $t \geq a$, $\mathcal{U}(t-a) = 1$, so

$$g(t)(1 - \mathcal{U}(t-a)) + h(t)\mathcal{U}(t-a) = g(t)(1-1) + h(t)(1) = h(t),$$

which also matches $f(t)$.

Similarly,

$$f(t) = \begin{cases} 0 & \text{if } 0 \leq t < a \\ g(t) & \text{if } a \leq t < b \\ 0 & \text{if } b \leq t \end{cases}$$

is

$$g(t)(\mathcal{U}(t-a) - \mathcal{U}(t-b)).$$

Let's examine why: when $0 \leq t < a$, $\mathcal{U}(t-a)$ and $\mathcal{U}(t-b)$ are both 0, so

$$g(t)(\mathcal{U}(t-a) - \mathcal{U}(t-b)) = g(t)(0-0) = 0.$$

When $a \leq t < b$, $\mathcal{U}(t-a) = 1$ and $\mathcal{U}(t-b) = 0$, so

$$g(t)(\mathcal{U}(t-a) - \mathcal{U}(t-b)) = g(t)(1-0) = g(t).$$

When $t \geq b$, $\mathcal{U}(t-a)$ and $\mathcal{U}(t-b)$ are both 1, so

$$g(t)(\mathcal{U}(t-a) - \mathcal{U}(t-b)) = g(t)(1-1) = 0.$$

Part 7: The Second Translation Theorem

Theorem 7.1. If $F(s) = \mathcal{L}\{f(t)\}$ and $a > 0$, then

$$\mathcal{L}\{f(t-a)\mathcal{U}(t-a)\} = e^{-as}F(s).$$

To compute $\mathcal{L}\{\mathcal{U}(t-a)\}$, take f to equal 1 and so $F(s) = \dfrac{1}{s}$. Then

$$\mathcal{L}\{\mathcal{U}(t-a)\} = \frac{e^{-as}}{s}.$$

Corollary 7.2. An alternate form of the Second Translation Theorem:

$$\mathcal{L}\{g(t)\mathcal{U}(t-a)\} = e^{-as}\mathcal{L}\{g(t+a)\}.$$

Corollary 7.3. Since the Second Translation Theorem tells us $\mathcal{L}\{f(t-a)\mathcal{U}(t-a)\} = e^{-as}F(s)$, applying $\mathcal{L}^{-1}$ on both sides of the equation gives us

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\mathcal{U}(t-a).$$

Example 7.4. Find the Laplace transform of $f(t) = 2(\mathcal{U}(t-2)-\mathcal{U}(t-4))-\mathcal{U}(t-4)$. First let's simplify $f(t)$ to $2\mathcal{U}(t-2) - 3\mathcal{U}(t-4)$. Then by the Theorem and linearity:

$$\mathcal{L}\{f(t)\} = 2\mathcal{L}\{\mathcal{U}(t-2)\} - 3\mathcal{L}\{\mathcal{U}(t-4)\} = \frac{2e^{-2s} - 3e^{-4s}}{s}.$$

Example 7.5. Consider $\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s+3}\right\}$. We can calculate this using the Inverse Second Translation Theorem (Corollary 7.3). First, we can ignore the e^{-s}, which gives us

$$\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} = e^{-3t}.$$

Then the Inverse Second Translation Theorem says to adjust for the e^{-s} by shifting our answer by 1 and multiplying by $\mathcal{U}(t-1)$. This gives us $e^{-3(t-1)}\mathcal{U}(t-1)$ as the answer.

Example 7.6. Consider $\mathcal{L}^{-1}\left\{\frac{se^{-\frac{\pi}{2}s}}{s^2+4}\right\}$. First let's ignore the $e^{-\frac{\pi}{2}}$ and calculate

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} = \cos(2t).$$

Then the Inverse Second Translation Theorem says to adjust for the $e^{-\frac{\pi}{2}}$ by shifting our result by $\frac{\pi}{2}$ and multiplying by $\mathcal{U}(t-\frac{\pi}{2})$. This gives us $\cos(2(t-\frac{\pi}{2}))\mathcal{U}(t-\frac{\pi}{2})$ as the answer.

Example 7.7. Solve $y' + 2y = f(t)$ such that $y(0) = 0$, where

$$f(t) = \begin{cases} t & \text{if } 0 \le t < 1 \\ 0 & \text{if } 1 \le t \end{cases}.$$

Begin by writing $f(t) = t(1-\mathcal{U}(t-1))$, or $t - t\mathcal{U}(t-1)$. Then by the alternate form of the Second Translation Theorem (Corollary 7.2),

$$\mathcal{L}\{t - t\mathcal{U}(t-1)\} = \frac{1}{s^2} - e^{-s}\mathcal{L}\{t+1\} = \frac{1}{s^2} - e^{-s}\left(\frac{1}{s^2} + \frac{1}{s}\right).$$

So by applying $\mathcal{L}$ to both sides of the DE, we get

$$sY(s) - y(0) + 2Y(s) = \frac{1}{s^2} - e^{-s}\left(\frac{1}{s^2} + \frac{1}{s}\right),$$

and by plugging in 0 for $y(0)$, we get

$$(s+2)Y(s) = \frac{1}{s^2} - e^{-s}\left(\frac{1}{s^2} + \frac{1}{s}\right).$$

Solving for $Y(s)$ gives us

$$Y(s) = \frac{1}{s^2(s+2)} - e^{-s}\left(\frac{1}{s^2(s+2)} + \frac{1}{s(s+2)}\right),$$

which requires a partial fraction decomposition for $\frac{1}{s^2(s+2)}$ and $\frac{1}{s(s+2)}$.

For $\frac{1}{s^2(s+2)}$:

$$\frac{1}{s^2(s+2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+2},$$

and so

$$1 = As(s+2) + B(s+2) + Cs^2.$$

Plugging in $s = 0$ gives

$$1 = 2B,$$

so $B = \frac{1}{2}$. Plugging in $s = -2$ gives

$$1 = 4C,$$

so $C = \frac{1}{4}$. Now

$$1 = As^2 + 2As + \frac{1}{2}s + 1 + \frac{1}{4}s^2,$$

and the s^2-terms give us the equation $A + \frac{1}{4} = 0$, so $A = -\frac{1}{4}$. Thus

$$\frac{1}{s^2(s+2)} = -\frac{1}{4s} + \frac{1}{2s^2} + \frac{1}{4(s+2)}.$$

Now for $\frac{1}{s(s+2)}$:

$$\frac{1}{s(s+2)} = \frac{A}{s} + \frac{B}{s+2},$$

and so

$$1 = A(s+2) + Bs.$$

Plugging in $s = 0$ gives us $A = \frac{1}{2}$. Plugging in $s = -2$ gives us $B = -\frac{1}{2}$. Thus

$$\frac{1}{s(s+2)} = \frac{1}{2s} - \frac{1}{2(s+2)}.$$

Putting it all together:

$$Y(s) = -\frac{1}{4s} + \frac{1}{2s^2} + \frac{1}{4(s+2)} + e^{-s}\left(-\frac{1}{4s} + \frac{1}{2s^2} + \frac{1}{4(s+2)} + \frac{1}{2s} - \frac{1}{2(s+2)}\right),$$

which simplifies to

$$Y(s) = -\frac{1}{4s} + \frac{1}{2s^2} + \frac{1}{4(s+2)} + e^{-s}\left(\frac{1}{4s} + \frac{1}{2s} - \frac{1}{4(s+2)}\right).$$

First let's take the inverse Laplace transform of the first bit: $-\frac{1}{4s}+\frac{1}{2s^2}+\frac{1}{4(s+2)}$. Following The Chart, the inverse transform is $-\frac{1}{4}+\frac{t}{2}+\frac{e^{-2t}}{4}$.

For the second bit, we will need the Inverse Second Translation Theorem (Corollary 7.3). Let's start by ignoring the e^{-s} and focus on the inverse Laplace transform of $\frac{1}{4s}+\frac{1}{2s}-\frac{1}{4(s+2)}$. Following The Chart, the inverse transform is $\frac{1}{4}+\frac{t}{2}-\frac{e^{-2t}}{4}$. The Inverse Second Translation Theorem says next we have to compensate for the e^{-s} by shifting our result by 1 and multiplying by $\mathcal{U}(t-1)$. So we get

$$\left(\frac{1}{4} + \frac{t-1}{2} - \frac{e^{-2(t-1)}}{4}\right)\mathcal{U}(t-1).$$

Putting everything together now, we get the answer

$$y(t) = -\frac{1}{4} + \frac{t}{2} + \frac{e^{-2t}}{4} + \left(\frac{1}{4} + \frac{t-1}{2} - \frac{e^{-2(t-1)}}{4}\right)\mathcal{U}(t-1).$$

Part 8: Deflection of a Beam

Consider a beam of length L. Think of the deflection of a beam y as a function of x. If the beam carries a load of $w(x)$ per unit length (so $\int_0^L w(x)\mathsf{d}x = \texttt{total weight}$), then we have the following DE for y:

$$EI\frac{\mathsf{d}^4 y}{\mathsf{d}x^4} = w(x),$$

where E is elasticity and I is moment of inertia.

Example 8.1. Suppose we have a 10 ft beam clamped at both ends (meaning $y(0) = y'(0) = y(10) = y'(10) = 0$), and that 10 lbs of weight are distributed uniformly across a 2 ft span across the center of the beam. So all of the eight is concentrated in the interval $[4, 6]$. Since the 10 lbs are distributed uniformly across 2 ft, that is 5 pounds per foot. The weight density function is

$$w(x) = \begin{cases} 0 & \text{if } 0 \le x < 4 \\ 5 & \text{if } 4 \le x \le 6 \\ 0 & \text{if } 6 < x \end{cases}.$$

We will find the deflection y of the beam, with E and I left unspecified. We can set up the Boundary Value Problem

$$EIy^{(4)} = 5(\mathcal{U}(x-4) - \mathcal{U}(x-6)),$$

with the boundary conditions $y(0) = y'(0) = y(10) = y'(10) = 0$. Applying $\mathcal{L}$ on both sides yields

$$EI(s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0)) = 5\left(\frac{e^{-4s}}{s} - \frac{e^{-6s}}{s}\right).$$

The boundary conditions tell us $y(0) = 0$ and $y'(0) = 0$, but tell us nothing about $y''(0)$ and $y'''(0)$, so we will call those c_1 and c_2 for now:

$$EI(s^4Y(s) - sc_1 - c_2) = 5\left(\frac{e^{-4s}}{s} - \frac{e^{-6s}}{s}\right).$$

Solving for $Y(s)$ gives us

$$Y(s) = \frac{5}{EI}\left(\frac{e^{-4s}}{s^5} - \frac{e^{-6s}}{s^5}\right) + \frac{c_1}{s^3} + \frac{c_2}{s^4}.$$

The Inverse Second Translation Theorem (Corollary 7.3) tells us how to compute $\mathcal{L}^{-1}\left\{\frac{e^{-4s}}{s^5}\right\}$ and $\mathcal{L}^{-1}\left\{\frac{e^{-6s}}{s^5}\right\}$. First ignore the exponential functions, so all we have is $\mathcal{L}^{-1}\left\{\frac{1}{s^5}\right\}$. Then we find the inverse Laplace transform, which is $\frac{x^4}{4!} = \frac{x^4}{24}$. Then we adjust for that e^{-4s} by shifting this result by 4 and multiplying by $\mathcal{U}(x-4)$; and we adjust for that e^{-6s} by shifting the $\frac{x^4}{24}$ by 6 and multiplying by $\mathcal{U}(x-6)$. So we get

$$\mathcal{L}^{-1}\left\{\frac{e^{-4s}}{s}\right\} = \frac{(x-4)^4}{24}\mathcal{U}(x-4)$$
$$\mathcal{L}^{-1}\left\{\frac{e^{-6s}}{s}\right\} = \frac{(x-6)^4}{24}\mathcal{U}(x-6)$$

And remember that The Chart says the $\mathcal{L}^{-1}\left\{\frac{c_1}{s^3}\right\} = \frac{c_1x^2}{2}$ and $\mathcal{L}^{-1}\left\{\frac{c_2}{s^4}\right\} = \frac{c_2x^3}{6}$. Putting it all together, we get

$$y(x) = \frac{5}{EI}\left(\frac{(x-4)^4}{24}\mathcal{U}(x-4) - \frac{(x-6)^4}{24}\mathcal{U}(x-6)\right) + \frac{c_1x^2}{2} + \frac{c_2x^3}{6}.$$

Now we need to solve for c_1 and c_2 by using the boundary conditions $y(10) = 0$ and $y'(10) = 0$. Thankfully, the step functions do not create trouble for us when we are solving: since are using $x = 10$, we can plug in 10 for x in $\mathcal{U}(x-4)$ and $\mathcal{U}(x-6)$. Since $\mathcal{U}(10-4) = 1$

and $\mathcal{U}(10-6)=1$, we can take the step functions to just be 1 (this also means that $\frac{\mathsf{d}}{\mathsf{d}x}\mathcal{U}(x-4)=\frac{\mathsf{d}}{\mathsf{d}x}1=0$, and $\frac{\mathsf{d}}{\mathsf{d}x}\mathcal{U}(x-6)=\frac{\mathsf{d}}{\mathsf{d}x}1=0$).

So y for $x>6$ is

$$y(x)=\frac{5}{EI}\left(\frac{(x-4)^4}{24}-\frac{(x-6)^4}{24}\right)+\frac{c_1x^2}{2}+\frac{c_2x^3}{6}$$

and y' for $x>6$ is

$$y'(x)=\frac{5}{EI}\left(\frac{(4(x-4)^3}{24}-\frac{4(x-6)^3}{24}\right)+c_1x+\frac{c_2x^2}{2}.$$

Plugging in 10 for x in $y(x)$, we get

$$0=\frac{5}{EI}\left(\frac{6^4}{24}-\frac{4^4}{24}\right)+50c_1+\frac{1000c_2}{6}.$$

Plugging in 10 for x in $y'(x)$, we get

$$0=\frac{5}{EI}\left(6^2-\frac{4^3}{6}\right)+10c_1+50c_2.$$

Simplifying everything, we get

$$0=\frac{650}{3EI}+50c_1+\frac{500c_2}{3}$$

and

$$0=\frac{380}{3EI}+10c_2+50c_3,$$

which results in $c_1=\frac{37}{3EI}$ and $c_2=\frac{5}{EI}$. This results in

$$y(x)=\frac{5}{24EI}((x-4)^4\mathcal{U}(x-4)-(x-6)^4\mathcal{U}(x-6))+\frac{37}{6EI}x^2-\frac{5}{6EI}x^3.$$

Part 9: Derivatives of Laplace Transforms

Theorem 9.1. Given $F(s)=\mathcal{L}\{f(t)\}$, then for $n=1,2,3,\ldots,$

$$\mathcal{L}\{t^nf(t)\}=(-1)^n\frac{\mathsf{d}^n}{\mathsf{d}s^n}F(s).$$

Example 9.2.

$$\mathcal{L}\{t\sin(2t)\}=-\frac{\mathsf{d}}{\mathsf{d}s}\mathcal{L}\{\sin(2t)\}=-\frac{\mathsf{d}}{\mathsf{d}s}\frac{2}{s^2+4}=\frac{4s}{(s^2+4)^2}.$$

Example 9.3.

$$\mathcal{L}\{t^2e^t\}=\frac{\mathsf{d}^2}{\mathsf{d}s^2}\mathcal{L}\{e^t\}=\frac{\mathsf{d}^2}{\mathsf{d}s^2}\frac{1}{s-1}=\frac{2}{(s-3)^3}.$$

Example 9.4.

$$\mathcal{L}\{te^t\sin(t)\} = -\frac{\mathsf{d}}{\mathsf{d}s}\mathcal{L}\{e^t\sin(t)\}.$$

Remember the First Translation Theorem (Theorem 3.1) says that $\mathcal{L}\{e^{at}f(t)\} = F(s-a)$. So

$$-\frac{\mathsf{d}}{\mathsf{d}s}\mathcal{L}\{e^t\sin(t)\} = -\frac{\mathsf{d}}{\mathsf{d}s}\frac{1}{(s-1)^2+1} = \frac{2(s-1)}{((s-1)^2+1)^2}.$$

Part 10: Convolution

Definition 10.1. Given f and g on $[0,\infty)$, the **convolution** of f and g is defined to be

$$f(t) * g(t) = \int_0^t f(\tau)g(t-\tau)\mathsf{d}\tau.$$

Note that $f(t) * g(t) = g(t) * f(t)$, you can use a u-substitution $u = t-\tau$ to get from one to the other.

Theorem 10.2. The Convolution Theorem: If f and g have are piecewise continuous on $[0,\infty)$ and of exponential order, then

$$\mathcal{L}\{f(t) * g(t)\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}.$$

Corollary 10.3. The inverse version of the Convolution Theorem is that if $F(s)$ and $G(s)$ are Laplace transforms of f and g, respectively, then

$$\mathcal{L}^{-1}\{F(s)G(s)\} = f(t) * g(t).$$

Example 10.4. What is $t * e^t$?

$$\int_0^t \tau e^{t-\tau}\mathsf{d}\tau = \int_0^t \tau e^t e^{\tau}\mathsf{d}\tau = e^t\int_0^t \tau e^{-\tau}.$$

Using integration by parts:

$$\begin{aligned} u &= \tau & \mathsf{d}v &= e^{-\tau}\mathsf{d}\tau \\ \mathsf{d}u &= \mathsf{d}\tau & v &= -e^{-\tau} \end{aligned}$$

we get

$$e^t\left(-\tau e^{-\tau}\Big|_0^t - \int_0^t -e^{-\tau}\mathsf{d}\tau\right) = e^t\left(-te^{-t} - e^{-\tau}\Big|_0^t\right) = e^t(-te^{-t} - e^{-t} + 1) = -t - 1 + e^t.$$

Example 10.5. We can use the Convolution Theorem to compute

$$\mathcal{L}\left\{\int_0^t \sin(\tau)\cos(t-\tau)\mathsf{d}\tau\right\}.$$

Since this is just $\mathcal{L}\{\sin(t) * \cos(t)\}$, Theorem 10.2 tells us

$$\mathcal{L}\left\{\int_0^t \sin(\tau)\cos(t-\tau)\mathsf{d}\tau\right\} = \mathcal{L}\{\sin(t)\}\mathcal{L}\{\cos(t)\} = \frac{1}{s^2+1}\cdot\frac{s}{s^2+1} = \frac{s}{(s^2+1)^2}.$$

Example 10.6. Find the Laplace transform $\mathcal{L}\left\{\int_0^t \cos(\tau)\mathsf{d}\tau\right\}$. Notice that $\int_0^t \cos(\tau)\mathsf{d}\tau$ is the convolution $\cos(t) * 1$. Thus

$$\mathcal{L}\left\{\int_0^t \cos(\tau)\mathsf{d}\tau\right\} = \mathcal{L}\{\cos(t) * 1\} = \mathcal{L}\{\cos(t)\} \cdot \mathcal{L}\{1\} = \frac{s}{s^2+1} \cdot \frac{1}{s} = \frac{1}{s^2+1}.$$

Example 10.7. Find $\mathcal{L}^{-1}\left\{\frac{1}{s(s+1)}\right\}$ using convolution.

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} * \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} = 1 * e^{-t}.$$

Using the definition of convolution, this is

$$\int_0^t e^{-\tau}\mathsf{d}\tau = -e^{-\tau}\Big|_0^t = -e^{-t} + 1.$$

Notice that convolution gives an alternative to using partial fraction decomposition!

Part 11: Periodic Functions

Definition 11.1. A function $f(t)$ is periodic with period P if $f(t-P) = f(t)$ for all t. For example, the trig functions $\sin(t)$ and $\cos(t)$ are periodic with period 2π.

Theorem 11.2. If $f(t)$ is a periodic, piecewise continuous function on $[0,\infty)$ of exponential order, with period P, then

$$\mathcal{L}\{f(t)\}(s) = \frac{1}{1-e^{-sP}} \int_0^P e^{-st} f(t)\mathsf{d}t.$$

Example 11.3. Since $\cos(t)$ is periodic with period 2π, $\mathcal{L}\{\cos(t)\} = \frac{1}{1-e^{-2\pi s}} \int_0^{2\pi} e^{-st}\cos(t)\mathsf{d}t.$

Part 12: The Dirac Delta

Definition 12.1. The unit impulse function δ_a is defined to be

$$\delta_a(t) = \begin{cases} 0 & \text{if } 0 \le t \le -a \\ \frac{1}{2a} & \text{if } -a \le t < a \\ 0 & \text{if } t \ge a \end{cases}.$$

Using step functions, we can rewrite this as

$$\delta_a(t) = \frac{1}{2a}(\mathcal{U}(t+a) - \mathcal{U}(t-a)).$$

Notice that

$$\int_{-\infty}^{\infty} \delta_a(t)\mathsf{d}t = \int_{-\infty}^{-a} 0\mathsf{d}t + \int_{-a}^{a} \frac{1}{2a}\mathsf{d}t + \int_{a}^{\infty} 0\mathsf{d}t = 1.$$

We can also shift the unit impulse function to the right by t_0: this gives us the function $\delta_a(t - t_0)$.

Definition 12.2. The Dirac Delta "function" is

$$\delta(t - t_0) = \lim_{a\to 0} \delta_a(t - t_0).$$

Some properties of $\delta(t - t_0)$ are:

1. $\delta(t - t_0) = \begin{cases} \infty & \text{if } t = t_0 \\ 0 & \text{if } t \neq t_0 \end{cases}$
2. $\displaystyle\int_{-\infty}^{\infty} \delta(t - t_0)\mathsf{d}t = \lim_{a\to 0} \int_{-\infty}^{\infty} \delta_a(t - t_0)\mathsf{d}t = 1.$

****Note that** the Dirac Delta function is not actually a function.

Theorem 12.3. We can define the Laplace transform as $\lim_{a\to 0} \mathcal{L}\{\delta_a(t - t_0)\}(s)$. Then

$$\mathcal{L}\{\delta(t - t_0)\} = e^{-st_0}.$$

In particular, if $t_0 = 0$,

$$\mathcal{L}\{\delta(t)\} = 1.$$

Example 12.4. Suppose we have a spring and mass system with mass $m = 1$, spring constant $k = 1$, and with an external force of $4\delta(t - 2\pi)$. Solve the IVP

$$y'' + y = 4\delta(t - 2\pi)$$

such that $y(0) = 0$, $y'(0) = 0$. Applying $\mathcal{L}$ to both sides gives

$$\mathcal{L}\{y'' + y\} = 4e^{-2\pi s}.$$

$$s^2Y(s) - sy(0) - y'(0) + Y(s) = 4e^{-2\pi s}.$$

Using 0 for $y(0)$ and 0 for $y'(0)$, we get

$$s^2Y(s) + Y(s) = 4e^{-2\pi s},$$

and solving for $Y(s)$ gives

$$Y(s) = \frac{4e^{-2\pi s}}{s^2 + 1}.$$

We can take the inverse Laplace transform by using the Inverse Second Translation Theorem (Corollary 7.3). First ignore the $e^{-2\pi s}$ and focus on $\mathcal{L}^{-1}\left\{\frac{4}{s^2+1}\right\}$, which is $4\sin(t)$. Then we adjust for the $e^{-2\pi s}$ by shifting this result by 2π and multiplying by $\mathcal{U}(t - 2\pi)$. Thus we get the answer

$$y = \mathcal{L}^{-1}\left\{\frac{4e^{-2\pi s}}{s^2 + 1}\right\} = 4\sin(t - 2\pi)\mathcal{U}(t - 2\pi).$$

Example 12.5. Solve the IVP $y'' + y = 4\delta(t-2\pi)$ such that $y(0) = 1$, $y'(0) = 0$. Applying $\mathcal{L}$ to both sides gives

$$\mathcal{L}\{y'' + y\} = 4e^{-2\pi s}.$$

$$s^2Y(s) - sy(0) - y'(0) + Y(s) = 4e^{-2\pi s}.$$

Using 1 for $y(0)$ and 0 for $y'(0)$, we get

$$Y(s)(s^2+1) - s = 4e^{-2\pi s}.$$

Solving for $Y(s)$ gives

$$Y(s) = \frac{4e^{-2\pi s}}{s^2+1} + \frac{s}{s^2+1}.$$

We already calculated $\mathcal{L}^{-1}\left\{\frac{4e^{-2\pi s}}{s^2+1}\right\} = 4\sin(t-2\pi)\mathcal{U}(t-2\pi)$ in the previous example. And $\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} = \cos(t)$, from The Chart. Thus

$$y = 4\sin(t-2\pi)\mathcal{U}(t-2\pi) + \cos(t).$$