Definition 1. A **numerical polynomial** is a polynomial $p \in k[t]$ such that for all $a \in \mathbb{Z}$, $p(a) \in \mathbb{Z}$.

Definition 2. Given a graded module M over a ring $S = k[x_0, \dots, x_n]$, the **Hilbert function** of M is $H_M : \mathbb{Z} \to \mathbb{N}$ such that

$$H_M(d) = \dim_k M_d.$$

Theorem Hilbert-Serre. Let M be a finitely-generated graded $S = k[x_0, \dots, x_n]$-module. Then there is a unique polynomial $h_M(t) \in \mathbb{Q}[t]$ such that $H_M(t) = h_M(t)$ for all $t \gg 0$. Furthermore, $\deg h_M = \dim V(\operatorname{Ann} M)$. This polynomial is the **Hilbert polynomial** of M.

Definition 3. Let $X \subseteq \mathbb{P}^n$ be an algebraic set. Then the **Hilbert polynomial** h_X of X is the Hilbert polynomial of its homogeneous coordinate ring $\Gamma(X)$ as a $k[x_0, \dots, x_n]$-module. The **degree** of X is $\dim(X)!$ times the leading coefficient of h_X.

Example 1. The Hilbert polynomial of $\mathbb{P}^n$ is $\binom{x+n}{n}$. The leading term of $H(x)$ is $\frac{x^n}{n!}$, so the dimension is n and the degree is $n!\frac{1}{n!} = 1$. (Do an example to show.)

Proposition 7.6.

1. If $X \subseteq \mathbb{P}^n$ and $X \neq \varnothing$, then the degree of X is a positive integer.
2. Let $X = X_1 \cup X_2$, where X_1 and X_2 have the same dimension r and $\dim(X_1 \cap X_2) < r$. Then $\deg X = \deg X_1 + \deg X_2$.
3. $\deg \mathbb{P}^n = 1$.
4. If $H \subseteq \mathbb{P}^n$ is a hypersurface whose ideal is generated by a homogeneous polynomial of degree d, then $\deg H = d$.

Example 2. Let $\nu_d : \mathbb{P}^n_k \hookrightarrow \mathbb{P}^N_k$ be the d-uple embedding, where $N = \binom{d+n}{n} - 1$. To find the degree of $X := \nu_d(\mathbb{P}^n_k) \subseteq \mathbb{P}^N_k$, we want to find the dimensions as k-vector spaces of each of the graded components $\Gamma(X)_m$ of $\Gamma(X) = k[x_0, \dots, x_N]/\mathfrak{I}(X)$.

Let us first look at the induced ring homomorphism $\nu_d^* : \Gamma(X) \to k[x_0, \dots, x_n]$. In this map $x_i + \mathfrak{I}(X)$ is sent to one of the ($N+1$ many) monomials of degree d in $k[x_0, \dots, x_n]$, for each $0 \leq i \leq N$.

So the question of how many monomials there are in $\Gamma(X)$ of degree m becomes a question of how many monomials there are in $k[x_0, \dots, x_n]$ of degree dm. The answer to this (with some combinatorial finagling) is $\binom{dm+n}{n}$.

Now we wish to find the Hilbert polynomial $H_X(m)$ of X. The Hilbert polynomial is a numerical polynomial whose input is m and whose output is $\dim_k \Gamma(X)_m$. In our case, $\dim_k \Gamma(X)_m = \binom{dm+n}{n}$. Observe that

$$\binom{dm+n}{n} = \frac{(dm+n)(dm+n-1)\cdots(1)}{(dm)!(n)!} = \frac{(dm+n)(dm+n-1)\cdots(dm+1)}{n!}$$

which when written out as a numerical polynomial has leading term $\frac{d^n m^n}{n!}$. (Recall $H_X(m) = \binom{dm+n}{n}$ is a numerical polynomial since it takes integers to integers.) Since the degree of X is $(\deg H_X)!$ times the leading coefficient of H_X, we know that $\deg X = n!\frac{d^n}{n!} = d^n$.

Example 3. Consider the 3-uple embedding $\nu_3 : \mathbb{P}^1_k \hookrightarrow \mathbb{P}^3_k$ defined to satisfy $\nu_3([a_0 : a_1]) = [a_0^3 : a_0^2 a_1 : a_0 a_1^2 : a_1^3]$. Then $\Gamma(X) = k[y_0, y_1, y_2, y_3]/(y_0 y_3 - y_1 y_2, y_0 y_2 - y_1^2, y_1 y_3 - y_2^2)$.

Clearly $\Gamma(X)_0 = k$. Now $\Gamma(X)_1$ is spanned by $\{y_0, y_1, y_2, y_3\}$, so $\dim_k \Gamma(X)_1 = 4$. Also $\Gamma(X)_2$ is spanned by $\{y_0^2, y_1^2, y_2^2, y_3^2, y_0 y_1, y_0 y_3, y_2 y_3\}$, so $\dim_k \Gamma(X)_2 = 7$. In general, $\dim_k \Gamma(X)_m = 3m + 1$, so $\deg X = (1)!3 = 3 = 3^1$.

The previous example is the twisted cubic.

Example 4. Consider the 2-uple embedding $\nu_2 : \mathbb{P}^2_k \hookrightarrow \mathbb{P}^5_k$, where we have the mapping $[a_0 : a_1 : a_2] \mapsto [a_0^2 : a_0 a_1 : a_0 a_2 : a_1^2 : a_1 a_2 : a_2^2]$. Let $X = \nu_2(\mathbb{P}^2_k) \subseteq \mathbb{P}^5_k$ and $\Gamma(X) = k[y_0, y_1, y_2, y_3, y_4, y_5]/\mathfrak{I}(X)$. Consider the induced ring homomorphism $\nu_2^* : \Gamma(X) \to k[x_0, x_1, x_2]$ satisfying $\overline{y_0} \mapsto x_0^2$, $\overline{y_1} \mapsto x_0 x_1$, $\overline{y_2} \mapsto x_0 x_2$, $\overline{y_3} \mapsto x_1^2$, $\overline{y_4} \mapsto x_1 x_2$, and $\overline{y_5} \mapsto x_2^2$.

Clearly $\Gamma(X)_0 = k$. Now $\Gamma(X)_1$ is generated by y_0 through y_5, and so $\dim_k \Gamma(X)_1 = 6$. Now the number of monomials in $\Gamma(X)$ of degree m is the number of monimials in $k[x_0, x_1, x_2]$ of degree $2m$, which is $\binom{2m+2}{2} = 2m^2 + 3m + 1$, so $\deg X = (2)!2 = 4 = 2^2$.

This is also called the **Veronese embedding**. It is useful for determining the conic through five points in projective space.

$$V(\det \begin{pmatrix} \nu_2(\vec{x}) & \nu_2(\vec{v}_1) & \nu_2(\vec{v}_2) & \nu_2(\vec{v}_3) & \nu_2(\vec{v}_4) & \nu_2(\vec{v}_5) \end{pmatrix}.$$

Proposition 1. The Hilbert function of $X \times Y \hookrightarrow \mathbb{P}^N$ via the Segre embedding is $h_X \cdot h_Y$.

Proof. The homogeneous coordinate ring $\Gamma(X \times Y) \cong \bigoplus_{d=0}^{\infty} (\Gamma(X)_d \otimes_k \Gamma(Y)_d)$.

This is because $(x_i, y_j) \mapsto x_i \otimes y_j$ induces a ring isomorphism. And $\dim(\Gamma(X)_d \otimes \Gamma(Y)_d) = \dim(\Gamma(X)_d) \cdot \dim(\Gamma(Y)_d)$. □

Example 5. Consider the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$. The Hilbert polynomial is $(x+1)^2 = x^2 + 2x + 1$. So the dimension is 2 and the degree is $2! \cdot 1 = 2$.

Example 6. Consider the Segre embedding $f : \mathbb{P}^1 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^5$. The Hilbert polynomial is $(x+1)(x^2 + 3x + 2)/2$. So the dimension is 3 and the degree is $3!/2 = 3$.

That means a general plane in $\mathbb{P}^5$ will meet $f(\mathbb{P}^1 \times \mathbb{P}^2)$ at three points!

Definition 4. The **arithmetic genus** of X is $\mathsf{g_a} = (-1)^r (h_X(0) - 1)$, where $r = \dim X$.

Example 7. The Hilbert polynomial of an elliptic curve $E \subseteq \mathbb{P}^2$ is $h_E(t) = 3t$. We can see the following table:

Degree	Generators	Dimension
1	x, y, z	3
2	$x^2, xy, xz, y^2, yz, z^2$	6
3	~~x^3~~	$10 - 1 = 9$
4	~~$x^3 \cdot x, y, z$~~	$15 - 3 = 12$

So the dimension of E is 1, the degree of E is 3, and the arithmetic genus of E is $(-1)^1(0-1) = 1$.

This matches the calculation for arithmetic genus given by the adjunction formula!

$$2g - 2 = E \cdot E + E \cdot K = 9 - 9 = 0.$$

Blowups and the double plane model? And weighted projective space!

Definition 4. For an algebraic set X, the **Hilbert series** for X is

$$HS_X(t) := \sum_{n=0}^{\infty} H_X(n)t^n.$$

Example 8. Let $X = \mathbb{P}(1, 2, 3)$ be a weighted projective plane. Then the Hilbert series for X is

$$\frac{1}{(1-t)(1-t^2)(1-t^3)} = 1 + t + 2t^2 + 3t^3 + 4t^4 + 5t^5 + 7t^6 + 8t^7 + 10t^8 + 12t^9 + 14t^{10} + \cdots .$$

A zero-scheme (or zero-cycle) is a configuration of points in $\mathbb{P}^n$. In 1986, Harbourne studied the Hilbert functions of certain zero-cycles.

Example 9. Consider the configuration of points $Z = \{(0, 0, 1), (0, 1, 0), (1, 0, 0)\} \subseteq \mathbb{P}^2$. The ideal defining these points is (xy, xz, yz), and so the Hilbert series is $1 + 3t + 3t^2 + \cdots$, and the Hilbert polynomial is 3.

Example 10. Now consider the configuration $Y = \{(0, 0, 1), (0, 1, 0), (0, 1, 1)\}$. This set of points has the defining ideal $(x, y^2z - z^3)$. The Hilbert series is $1 + 2t + 3t^2 + 3t^3 + \cdots$, and the Hilbert polynomial is 3.

Definition 5. A 0-cycle $Z \subseteq \mathbb{P}^2$ is **generic** if its Hilbert function is

$$H_Z(t) = \min\left\{\binom{t+2}{2}, h_Z(t)\right\}.$$

So the three points in Example 9 are generic, but in Example 10 they are not.

Masanori Koitabashi: If you blow up $\mathbb{P}^2$ at 7 and 8 generic points, the automorphism group of the resulting rational surface is C_2, generated by the Geiser and Bertini involutions, respectively.