Definition 1. Let C be a curve in $\mathbb{P}^2_k$ given by a homogeneous polynomial f. Then the **Hesse derivative** इC is the vanishing locus of the polynomial

$$\begin{vmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{xy} & f_{yy} & f_{yz} \\ f_{xz} & f_{yz} & f_{zz} \end{vmatrix}.$$

Proposition 1. Let $E \subseteq \mathbb{P}^2_k$ be an elliptic curve, $k = \mathbb{C}$. Then there exist an $a, b \in k$ such that $E \cong V(y^2z - x^3 + axz^2 + bz^3)$. Such a curve is called **Weierstrass form**.

Definition 2. Let $E \subseteq \mathbb{P}^2_k$ be an elliptic curve in Weierstrass form. Then the j-invariant of E is

$$1728 \cdot \frac{4a^3}{4a^3 + 27b^2}.$$

The j-invariant is an isomorphism invariant of elliptic curves.

Theorem K–. Let E be an elliptic curve with j-invariant j. Then the j-invariant of इE is

$$H(j) = \frac{(6912 - j)^2}{27j^2}.$$

Of particular interest are the j-invariants that are periodic over H.

For instance, the following nine numbers satisfy $H^2(j) = j$.

$$j = 1728$$
$$j = \frac{3456}{7}\left(-1 - 3i\sqrt{3}\right)$$
$$j = \frac{3456}{7}\left(-1 + 3i\sqrt{3}\right)$$
$$j = 3456\left(5 - 3\sqrt{3}\right)$$
$$j = 3456\left(3\sqrt{3} + 5\right)$$
$$j = -5184i\sqrt{3} - \frac{1}{2}\sqrt{-\frac{4514807808}{13} - \frac{1}{13}644972544i\sqrt{3}} + 1728$$
$$j = -5184i\sqrt{3} + \frac{1}{2}\sqrt{-\frac{4514807808}{13} - \frac{1}{13}644972544i\sqrt{3}} + 1728$$
$$j = 5184i\sqrt{3} - \frac{1}{2}\sqrt{-\frac{4514807808}{13} + \frac{644972544i\sqrt{3}}{13}} + 1728$$
$$j = 5184i\sqrt{3} + \frac{1}{2}\sqrt{-\frac{4514807808}{13} + \frac{644972544i\sqrt{3}}{13}} + 1728.$$

Theorem K–. Let E be an elliptic curve where with j-invariant j periodic under H. Then there is an $n \in \mathbb{N}$ such that इ$^nE = E$.

For example, if $j(E) = 1728$, then $\text{ह}^2 E = E$. But if $j(E) = \frac{3456}{7}\left(-1 - 3i\sqrt{3}\right)$, then $\text{ह}^3 E = E$.

Definition 3. Let E be an elliptic curve. The **Hesse pencil** $\mathcal{P}(E)$ generated by E is spanned by E and $\text{ह}E$.

The key: G_{216} permutes the isomorphic fibers of the Hesse pencil.

This can be known by studying the automorphism group of the Hesse pencil (a subgroup of $\mathrm{Aut}(\mathbb{P}^2)$), called the Hesse group. Along with the fact that

$$\text{ह}\alpha(E) = \alpha(\text{ह}E) \text{ for all } \alpha \in \mathrm{Aut}(\mathbb{P}^2).$$

The Hesse group is

$$G_{216} \cong \Gamma \rtimes \Delta$$

where $\Gamma \cong (\mathbb{Z}/3\mathbb{Z})^2$ preserves each fibre and $\Delta \cong \mathrm{SL}_2(\mathbb{F}_3)$ acts on Γ by linear representation. Specifically,

$$\Delta = \left\langle \alpha = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \zeta_3 & \zeta_3^2 \\ 1 & \zeta_3^2 & \zeta_3 \end{pmatrix}, \beta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta_3 & 0 \\ 0 & 0 & \zeta_3 \end{pmatrix} \right\rangle \leq \mathrm{Aut}(\mathbb{P}^2) = \mathrm{PGL}(3)$$

and

$$\Gamma \rtimes \langle \alpha^2 \rangle = \ker(G_{216} \to \mathrm{Aut}(\mathbb{P}^1)).$$

Theorem K–. The number of orbits of size n under H is

$$\frac{\sum_{d|n} \mu(d) 3^{n/d}}{n}.$$

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