New Perspectives on Geproci-ness

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What is Geproci?

Definition

A finite set Z in $\mathbb{P}^n_k$ is **geproci** if the projection $\overline{Z}$ of Z from a general point P to a hyperplane is a complete intersection in $\mathbb{P}^{n-1}_k$.

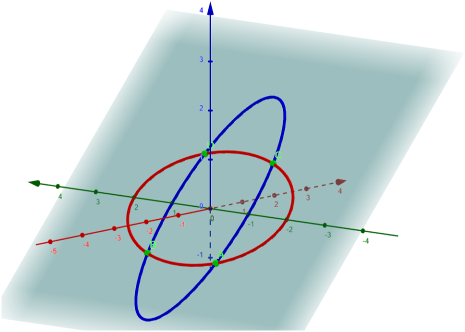
Geproci stands for **ge**neral **pro**jection is a **c**omplete **i**ntersection.

The only nontrivial examples known are for $n = 3$. In this case a hyperplane is a plane H. A reduced set of points in a plane is a complete intersection if it is the transverse intersection of two algebraic curves, like this.

For $\#Z = ab$ $(a \leq b)$, Z is (a, b)-geproci if $\overline{Z}$ is the intersection of a degree a curve and a degree b curve.

What We Know: Coplanar Points

A set of coplanar points can be geproci only if they are already a complete intersection in the plane they're on.



What We Know: Grids

Definition

A **grid** in $\mathbb{P}^3$ is a set of points that form the intersection of two families of mutually-skew lines.

Every grid is geproci, and the projection of the points of a grid is a complete intersection of two unions of lines.

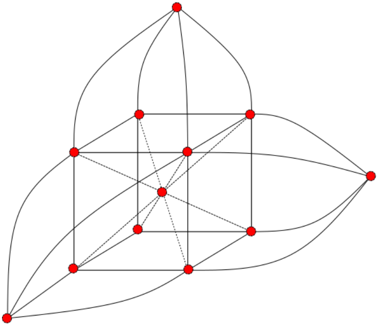
Grids and coplanar points are the trivial cases of geproci-ness.

An (a, b)-grid with $3 \leq a \leq b$ is always a set of points on a smooth quadric.

What We Know: D_4

D_4 is a set of 12 points and 16 3-rich lines. It is $(3,4)$-geproci and the smallest non-trivial geproci set in characteristic 0.

D_4 is a *half-grid*. It is also the only non-trivial $(3,b)$-geproci set where $b \geq 3$ in characteristic 0.



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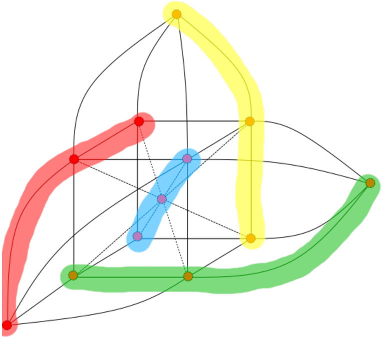
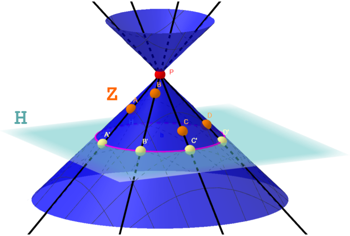
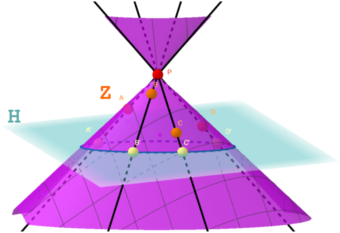


Cones and Geproci

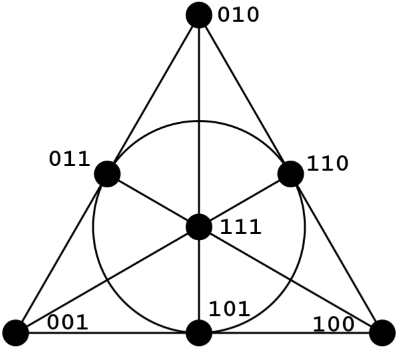
It is of interest when a cone through Z whose vertex is a general point P, and which meets H in a curve containing the projected image of Z. When Z is (a, b)-geproci, there are two such cones, of degrees a, b.





Geometry in Positive Characteristic

Geometry gets weird in positive characteristic! You may already be familiar with $\mathbb{P}^2_{\mathbb{Z}/2\mathbb{Z}}$, aka the **Fano Plane**.



Cones in $\mathbb{P}^3_{\mathbb{F}_q}$ of degree $a = q + 1$

It turns out geprociness is very natural in positive characteristic.

Note that $\#\mathbb{P}^3_{\mathbb{F}_q} = \dfrac{q^4 - 1}{q - 1} = q^3 + q^2 + q + 1 = (q+1)(q^2+1)$.

There is a degree $q + 1$ cone containing $\mathbb{P}^3_{\mathbb{F}_q}$ whose vertex is at a general point $P = (a, b, c, d) \in \mathbb{P}^3_k$, $k = \overline{\mathbb{F}}_q$. This cone is given by

$$\begin{aligned} (c^q d - cd^q)(x^q y - xy^q) - (b^q d - bd^q)(x^q z - xz^q) \\ +(b^q c - bc^q)(x^q w - xw^q) + (a^q d - ad^q)(y^q z - yz^q) \\ -(a^q c - ac^q)(y^q w - yw^q) + (a^q b - ab^q)(z^q w - zw^q) \end{aligned}$$

Spreads in $\mathbb{P}^3_{\mathbb{F}_q}$

Is there a cone of degree $b = q^2 + 1$? There is!

Each line of $\mathbb{P}^3_{\mathbb{F}_q}$ contains $q + 1$ points. Can $\mathbb{P}^3_{\mathbb{F}_q}$ be partitioned by $q^2 + 1$ mutually-skew lines? Yes! Such a partition is called a **spread**.

The join of a general point P with the lines of a spread gives the desired cone of degree $q^2 + 1$.

Existence of Spreads

Theorem (Bruck and Bose '63)

Let $\mathbb{P}^{2t-1}_{\mathbb{F}_q}$ be an odd-dimensional projective space over a field $\mathbb{F}_q$ of size q, where q is a power of a prime. Then there exists a spread in $\mathbb{P}^{2t-1}_{\mathbb{F}_q}$.

Proof.

Let $L = \mathbb{F}_{q^{2t}}$, $K = \mathbb{F}_{q^t}$, and $F = \mathbb{F}_q \subseteq K \subseteq L$. Then L is a 2-dimensional vector space over K, and K is a t-dimensional vector space over F. Hence, $\mathbb{P}^{2t-1}_{\mathbb{F}_q} = \mathbb{P}(L/F)$ and $\mathbb{P}^1_{\mathbb{F}_{q^t}} = \mathbb{P}(L/K)$. The set S of all 1-dimensional vector subspaces of L over K is also a set of t-dimensional vector subspaces of L over F. And S is simultaneously a spread of $\mathbb{P}^1_K$ and a spread of $\mathbb{P}^{2t-1}_F$. □

A Theorem

Theorem (K–)

The set of points $\mathbb{P}^3_{\mathbb{F}_q}$ is $(q+1, q^2+1)$-geproci in $\mathbb{P}^3_k$, where k is an algebraically closed field containing $\mathbb{F}_q$.

Note when $q = 2$, we get a non-trivial $(3, 5)$-geproci set! These cannot happen in characteristic 0.

Partial Spreads

Definition

A **partial spread** of $\mathbb{P}^3_{\mathbb{F}_q}$ with deficiency d is a set of q^2+1-d mutually-skew lines. A **maximal partial spread** is a partial spread of positive deficiency that is not contained in any larger partial spread.

Theorem (K–)

The complement of a maximal partial spread of deficiency d is a non-trivial $\{q+1,d\}$-geproci set. Furthermore, when $d>q+1$, the complement is a non-trivial non-half-grid.

In 1965, Dale Mesner provided a lower bound for the size of the deficiency for maximal partial spreads at $\sqrt{q}+1\leq d$. Glynn provided an upper bound of $d\leq(q-1)^2$.

The field $\mathbb{F}_7$ and Gorenstein Configurations

The maximal partial spreads in $\mathbb{P}^3_{\mathbb{F}_7}$ have been classified by Soicher in 2000. They all comprise 45 lines, and their complements are configurations of 40 points.

Each complement is $(5, 8)$-geproci and is a non-half-grid. Furthermore, at least four of the fifteen are different up to projective equivalence and are Gorenstein! The four configurations I tested so far have stabilizers in $PGL(4, 7)$ of different sizes (10, 20, 60, and 120) and so are not projectively equivalent.

In characteristic 0, only one non-trivial Gorenstein configuration is known up to projective equivalence, also a configuration of 40 points.

Infinitely-Near Points

Definition

Let X be an algebraic variety and let $P \in X$. The point Q is **infinitely-near** P if Q is on the exceptional locus of the blowup of X at P. (Intuitively, Q is a tangent direction at P.)

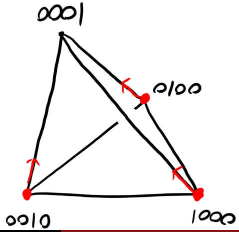
Abuse of notation: Technically, $Q \in \mathsf{BL}_P(X)$, but we will be speaking of infinitely-near points as if they were points of X itself.

Geproci With Infinitely-Near Points

Theorem (K–)

Let char $k = 2$. *Let* $Z = \{(1,0,0,0) \times 2, (0,1,0,0) \times 2, (0,0,1,0) \times 2\}$ *(where* $p_i \times 2$ *represents an ordinary point* $p_i \in \mathbb{P}^3_k$ *and a point* q_i *infinitely near* p_i*), with the infinitely-near point at each ordinary point corresponding to the tangent along the line through* p_i *and* $(0,0,0,1)$.

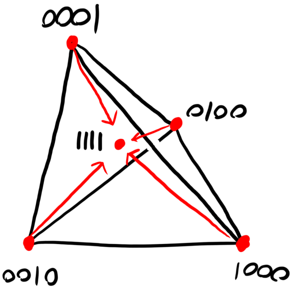
Then Z *is a* $(2,3)$*-geproci half-grid.*



Another Example

Theorem (K–)

Let $Z = \{(1,0,0,0) \times 2, (0,1,0,0) \times 2, (0,0,1,0) \times 2, (0,0,0,1) \times 2, (1,1,1,1)\}$, *which each infinitely-near point corresponding to the line containing* $(1,1,1,1)$. *Then* Z *is a* $(3,3)$*-geproci. It is a non-trivial non-half-grid.*



Future Problems

1. Do infinitely-near points provide new examples of non-trivial geproci sets in characteristic 0?
2. Does taking higher-order infinitely-near points provide new examples of geproci sets?
3. Do **maximal partial spreads** provide new examples of geproci sets that work in characteristic 0?
4. Can geproci sets give new results on spreads?