

1 Lie algebras notes

From Peter Woit's notes. A **Lie group** is a differentiable manifold with a group law. A **Lie algebra** is the tangent space of a Lie group at the identity. Usually Lie algebras are denoted in a fraktur font: e.g. SL_2 is a Lie group and $\mathfrak{sl}_2$ is its Lie algebra.

Definition 2. For G a Lie group of $n \times n$ invertible matrices, the Lie algebra $\mathfrak{g}$ is the space of $n \times n$ matrices X such that $e^{tX} \in G$ for $t \in \mathbb{R}$.

Example 1. Let $G = \mathrm{SL}_2(\mathbb{R})$. Then

$$G = \left\{ \begin{pmatrix} x & y \\ z & w \end{pmatrix} : xw - yz = 1 \right\}.$$

Then the tangent space at the identity $I_2 \in G$ is defined by $x + w = 2$, so

$$\mathfrak{g} = \left\{ \begin{pmatrix} x & y \\ z & w \end{pmatrix} : x + w = 2 \right\}$$

This is not actually a vector space (it is an affine space), so I think we have to shift to the origin and instead get the hyperplane defined by $x + w = 0$. Thus $\mathfrak{g}$ is the space of all traceless matrices.

Now we wish to show that if $\mathrm{tr}(X) = 0$, the e^X is invertible. In fact, we can show that $\det(e^X) = e^{\mathrm{tr}(X)}$ in general. First assume X is diagonalizable: then $X = P \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} P^{-1}$, so $\mathrm{tr}(X) = a + b$, since similar matrices have the same trace. Then

$$e^X = P \begin{pmatrix} e^a & 0 \\ 0 & e^b \end{pmatrix} P^{-1},$$

so $\det(e^X) = e^a * e^b = e^{a+b} = e^{\mathrm{tr}(X)}$. Since diagonalizable matrices are dense in the space of all matrices, (and $\exp$, $\det$, and tr are continuous functions), this carries over to all matrices. So in particular, if $\mathrm{tr}(X) = 0$, then $\det(e^{tX}) = e^{\mathrm{tr}(tX)} = e^0 = 1$, so $e^{tX} \in G$.

It is worth going into why exactly the Lie algebra as defined is a vector space. Let $X, Y \in \mathfrak{g}$, then by definition $e^X, e^Y \in G$, and so $e^{X+Y} = e^X e^Y \in G$ since G is a group. (I guess $\exp$ matrices always commute with each other? Claim: $\{e^X : X \in \mathbb{R}^{n \times n}\}$ is an abelian group.)

Definition 3. The **adjoint representation** ($Ad, \mathfrak{g}$) is given by the homomorphism

$$Ad : g \in G \rightarrow Ad(g) \in GL(\mathfrak{g})$$

where $Ad(g)$ acts on $X \in \mathfrak{g}$ by

$$(Ad(g))(X) = gXg^{-1}.$$

To show this is well-defined, one needs to check that $gXg^{-1} \in \mathfrak{g}$ when $X \in \mathfrak{g}$, but this can be shown using the identity

$$e^{tgXg^{-1}} = ge^{tX}g^{-1}$$

which implies that $e^{tgXg^{-1}} \in G$ if $e^{tX} \in G$. To check this identity, expand the exponential

$$e^{tgXg^{-1}} = \sum_{k=0}^{\infty} \frac{(tgXg^{-1})^k}{k!}$$

and use

$$(gXg^{-1})^k = gX^kg^{-1}.$$

Definition 4. The **Lie bracket** operation on $\mathfrak{g}$ is the bilinear anti-symmetric map given by the commutator of matrices

$$[\cdot, \cdot] : (X, Y) \in \mathfrak{g} \times \mathfrak{g} \rightarrow [X, Y] = XY - YX \in \mathfrak{g}.$$

Let us make sure this is well-defined; i.e., it takes values in $\mathfrak{g}$.

Proof. Since $X \in \mathfrak{g}$, we have $e^{tX} \in G$ and we can act on $Y \in \mathfrak{g}$ by the adjoint representation

$$Ad(e^{tX})Y = e^{tX}Ye^{-tX} \in \mathfrak{g}.$$

As t varies, this gives us a parametrized curve in $\mathfrak{g}$. Its velocity vector will also be in $\mathfrak{g}$ (since $\mathfrak{g}$ is a hyperplane), so

$$\frac{d}{dt}(e^{tX}Ye^{-tX}) = Xe^{tX}Ye^{-tX} - e^{tX}YXe^{-tX}$$

by the product rule. Evaluating this at $t = 0$ yields

$$XY - YX$$

which thus must be in $\mathfrak{g}$ (again, because $\mathfrak{g}$ is a hyperplane and so contains all velocity vectors of paths contained within it). $\square$

Note: the general form for a 2×2 traceless non-diagonalizable matrix is $\begin{pmatrix} ab & b^2 \\ -a^2 & -ab \end{pmatrix}$ with eigenvector $\begin{pmatrix} b \\ -a \end{pmatrix}$. Since this matrix is nilpotent, its exponential is just

$$\exp \begin{pmatrix} ab & b^2 \\ -a^2 & -ab \end{pmatrix} = \begin{pmatrix} ab & b^2 \\ -a^2 & -ab \end{pmatrix} + I_2 = \begin{pmatrix} ab+1 & b^2 \\ -a^2 & -ab+1 \end{pmatrix}.$$

For this matrix to be orthogonal, we would need $ab^3 + b^2 - a^3b - a^2 = 0$, so $(b^2 - a^2)(ab + 1) = 0$. So either $a = b$, $-a = b$, or $ab = -1$. In the former case, we have

$$\begin{pmatrix} ab & b^2 \\ -a^2 & -ab \end{pmatrix} = \begin{pmatrix} a^2 & a^2 \\ -a^2 & -a^2 \end{pmatrix} = a^2 \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}.$$

In the second case, we have

$$\begin{pmatrix} ab & b^2 \\ -a^2 & -ab \end{pmatrix} = \begin{pmatrix} -a^2 & a^2 \\ -a^2 & a^2 \end{pmatrix} = a^2 \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}.$$

In the last case, we have

$$\begin{pmatrix} ab & b^2 \\ -a^2 & -ab \end{pmatrix} = \begin{pmatrix} -1 & a^{-2} \\ -a^2 & 1 \end{pmatrix}.$$

So the exponentials are $\begin{pmatrix} a^2 + 1 & a^2 \\ -a^2 & -a^2 + 1 \end{pmatrix}$, $\begin{pmatrix} -a^2 + 1 & a^2 \\ -a^2 & a^2 + 1 \end{pmatrix}$, and $\begin{pmatrix} 0 & a^{-2} \\ -a^2 & 2 \end{pmatrix}$

1.1 Lie algebras of the orthogonal and unitary groups

We have $UU^\dagger = 1$. Write $U = e^{tX}$, for some parameter t and X a matrix in the Lie algebra. Since the transpose of a product of matrices is the order-reversed product of their transposes, and the complex conjugate of a product of matrices is the product of the complex conjugates of the matrices, one has

$$(e^{tX})^\dagger = e^{tX^\dagger}.$$

The condition $UU^\dagger = 1$ becomes

$$e^{tX}e^{tX^\dagger} = 1.$$

Taking the derivative of this equation gives

$$e^{tX}X^\dagger e^{tX^\dagger} + X e^{tX}e^{tX^\dagger} = 0,$$

and evaluating at $t = 0$ yields

$$X + X^\dagger = 0.$$

So the matrices of the Lie algebra must be skew-adjoint.

1.2 $\mathfrak{so}(n)$ ($K = \mathbb{R}$)

Recall that the orthogonal group $O(n)$ has the condition $U^\top = U^{-1}$; we will restrict to $SO(n)$ in which we add the condition $\det U = 1$, with elements that can be written as

$$U = e^{tX}.$$

We saw above that X must be skew-symmetric. So in the case $G = SO(n)$, the Lie algebra $\mathfrak{so}(n)$ is the space of skew-symmetric real matrices. The dimension of such a space is

$$1 + 2 + \cdots + (n-1) = \frac{n^2 - n}{2}$$

and a basis will be given by the matrices ε_{jk} with $1 \leq j, k \leq n$ and $j < k$ defined by

$$(\varepsilon_{jk})_{\ell m} = \begin{cases} -1 & j = \ell, k = m, \\ 1 & j = m, k = \ell, \\ 0 & . \end{cases}$$

For example, say $n = 2$, then we have

$$\varepsilon_{12} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

For $n = 3$, we have

$$\varepsilon_{12} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\varepsilon_{13} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

and

$$\varepsilon_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

1.3 $\mathfrak{u}(n)$, ($K = \mathbb{C}$)

For the unitary group, we have that X must be skew-adjoint (a.k.a. skew-Hermitian), where $X^\dagger = -X$. So the Lie algebra $\mathfrak{u}(n)$ is the space of skew-adjoint $n \times n$ complex matrices, together with the bilinear, antisymmetric Lie bracket. Note that $\dim_{\mathbb{R}} \mathfrak{u}(n) = n^2$, so $\mathfrak{u}(n)$ is a real vector space of dimension n^2 , but it is NOT a space of real $n \times n$ matrices. For example, $\mathfrak{u}(2)$ is $\mathbb{R}$ -generated by the four matrices

$$\begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & i \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

In fact, $\mathfrak{u}(n)$ is not a vector space over $\mathbb{C}$ because multiplying by i interferes with the skew-Hermitian property.

1.4 Lie algebra representations

Definition 5. A **Lie algebra representation** of a Lie algebra $\mathfrak{g}$ on an n -dimensional complex vector space V is given by a real-linear map

$$\phi : \mathfrak{g} \rightarrow \mathfrak{gl}(n, \mathbb{C}) = \mathbb{C}^{n \times n}$$

satisfying

$$\phi([X, Y]) = [\phi(X), \phi(Y)].$$

Such a representation is called **unitary** if its image is $\mathfrak{u}(n)$, i.e., if $\phi(X)^\dagger = -\phi(X)$.

More concretely, given a basis $X_1, \dots, X_d$ of a Lie algebra $\mathfrak{g}$ of dimension d with “structure constants” c_{jkl} , a representation is given by a choice of d complex $n \times n$ matrices $\phi(X_j)$ satisfying

$$[\phi(X_j), \phi(X_k)] = \sum_{\ell=1}^d c_{jkl} \phi(X_\ell).$$

These structure constants are real numbers defining

$$[X_j, X_k] = \sum_{\ell=1}^d c_{jkl} X_\ell.$$

Since the Lie bracket is bilinear, it is sufficient to just know the structure constants between each pair of basis elements to be able to calculate the Lie bracket for any pair of matrices. Furthermore, since the Lie bracket is antisymmetric, $c_{jkl} = -c_{kjl}$ for all j, k .

Now consider the Lie group representation homomorphism $\pi : G \rightarrow GL(n, \mathbb{C})$. We can construct a linear map

$$\pi' : \mathfrak{g} \rightarrow \mathbb{C}^{n \times n}$$

where

$$\pi'(X) = \frac{d}{dt}(\pi(e^{tX}))|_{t=0}$$

satisfies

1. $\pi(e^{tX}) = e^{t\pi'(X)}$.
2. For $g \in G$, $\pi'(gXg^{-1}) = \pi(g)\pi'(X)\pi(g)^{-1}$.
3. π' is a Lie algebra homomorphism: commutes with the Lie bracket.

Proof. 1. We have

$$\begin{aligned} \frac{d}{dt}\pi(e^{tX}) &= \frac{d}{ds}\pi(e^{(t+s)X})|_{s=0} \\ &= \frac{d}{ds}\pi(e^{tX}e^{sX})|_{s=0} \\ &= \frac{d}{ds}\pi(e^{tX})\pi(e^{sX})|_{s=0} \\ &= \pi(e^{tX})\pi'(X) \end{aligned}$$

So $f(t) = \pi(e^{tX})$ satisfies the differential equation of the form

$$f'(t) = \pi'(X)f(t).$$

This is an exponential differential equation with solution $f(t) = e^{t\pi'(X)} + C$. Since $\pi(e^{tX})$ and $e^{t\pi'(X)} + 0$ both satisfy the initial condition $\pi(I_n) = I_n$ for $t = 0$, we have that the two must be equal.

2. Note that

$$\pi'(gXg^{-1}) = \frac{d}{dt}\pi(e^{tgXg^{-1}})|_{t=0} = \frac{d}{dt}\pi(ge^{tX}g^{-1})|_{t=0} = \pi(g)\frac{d}{dt}\pi(e^{tX})|_{t=0}\pi(g^{-1}) = \pi(g)\pi'(X)\pi(g)^{-1}.$$

3. First we will show π' is linear. Let $X, Y \in \mathfrak{g}$. Then

$$\begin{aligned}\pi'(X + Y) &= \frac{d}{dt}\pi(e^{t(X+Y)})|_{t=0} = \frac{d}{dt}\pi(e^{tX})\pi(e^{tY})|_{t=0} \\ &= \pi(e^{tX}) * \left[\frac{d}{dt}\pi(e^{tY}) \right] |_{t=0} + \left[\frac{d}{dt}\pi(e^{tX}) \right] * \pi(e^{tY})|_{t=0} = I_n * \pi'(Y) + \pi'(X) * I_n = \pi'(X) + \pi'(Y).\end{aligned}$$

Now let $r \in \mathbb{R}$. Then since π is a group homomorphism,

$$\pi'(rX) = \frac{d}{dt}\pi(e^{t(rX)})|_{t=0} = \frac{d}{dt}\pi(e^{tX})^r|_{t=0} = r\pi(e^{tX})^{r-1} * \frac{d}{dt}\pi(e^{tX})|_{t=0} = r\pi'(X).$$

Now we must show that π' commutes with the Lie bracket. We wish to show

$$\pi'([X, Y]) = [\pi'(X), \pi'(Y)]$$

where in $\mathbb{C}^{n \times n}$ the Lie bracket is $[\pi'(X), \pi'(Y)] = \pi'(X)\pi'(Y) - \pi'(Y)\pi'(X)$. Recall from earlier that

$$[X, Y] = \frac{d}{dt}(e^{tX}Ye^{-tX})|_{t=0},$$

so

$$\begin{aligned}\pi'([X, Y]) &= \pi'\left(\frac{d}{dt}(e^{tX}Ye^{-tX})|_{t=0}\right) \\ &= \frac{d}{dt}\pi'(e^{tX}Ye^{-tX})|_{t=0} \text{ (by linearity)} \\ &= \frac{d}{dt}(\pi(e^{tX})\pi'(Y)\pi(e^{-tX}))|_{t=0} \text{ (by 2.)} \\ &= \frac{d}{dt}(e^{t\pi'(X)}\pi'(Y)e^{-t\pi'(X)})|_{t=0} \text{ (by 1.)} \\ &= [\pi'(X), \pi'(Y)].\end{aligned}$$

□

For any Lie group G where is a distinguished representation called the adjoint representation (Ad, G) . The corresponding Lie algebra representation is also called the adjoint representation, but written as $(Ad', \mathfrak{g}) = (ad, \mathfrak{g})$. From the fact that

$$Ad(g)(Y) = gYg^{-1},$$

we have

$$Ad(e^{tX})(Y) = e^{tX}Ye^{-tX}$$

and differentiate with respect to t to get the Lie algebra representation

$$ad(X)(Y) = \frac{d}{dt}(e^{tX}Ye^{-tX})|_{t=0} = [X, Y].$$

So $ad(X)$ can be written as $[X, \cdot]$. The Lie algebra homomorphism property says

$$ad([X, Y]) = ad(X) \circ ad(Y) - ad(Y) \circ ad(X),$$

thus for $Z \in g$ we have

$$[[X, Y], Z] = [X, [Y, Z]] - [Y, [X, Z]].$$

This is called the Jacobi identity. An **abstract Lie algebra** is a vector space with the bilinear operation $[\cdot, \cdot]$ known as the Lie bracket that satisfies the antisymmetric property and the Jacobi identity.

It is worth noting that if f is a linear function, then $f(g'(t)) = [f(g(t))]'$. This is because since f is linear, f is continuous, so limits commute with f . Then

$$f(g'(t)) = f\left(\lim_{\varepsilon \rightarrow 0} \frac{g(t + \varepsilon) - g(t)}{\varepsilon}\right) = \lim_{\varepsilon \rightarrow 0} \frac{f(g(t + \varepsilon)) - f(g(t))}{\varepsilon} = [f(g(t))]'.$$