

Definition 1. Let V be a finite-dimensional vector space over a field K , $\text{char} K \neq 2$. Then a **quadratic form** is a function $Q : V \rightarrow K$ such that $Q(\lambda v) = \lambda^2 Q(v)$ for all $v \in V$ and $\lambda \in K$, and $B : V \times V \rightarrow K$ given by $B(u, v) = Q(u + v) - Q(u) - Q(v)$ is bilinear.

Remark 1. Given any n -dimensional vector space V/K and quadratic form $Q : V \rightarrow K$, there exists an $n \times n$ symmetric matrix A such that

$$Q(v) = v^T A v$$

for all $v \in V$.

Definition 2. Let V be a finite-dimensional vector space over a field K , $\text{char} K \neq 2$. Let $Q : V \rightarrow K$ be a quadratic form. Then the **Clifford algebra determined by V and Q** is

$$\text{Cl}(V, Q) := \bigoplus_{n=0}^{\infty} V^{\otimes n} / \langle v \otimes v - Q(v) \rangle.$$

Note that Q is equivalent to some symmetric matrix A . Since A is symmetric, it is orthogonally diagonalizable. That is, there exists an orthogonal matrix $U \in O(n)$ such that

$$A = U \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{pmatrix} U^T.$$

Thus we may pick a basis $\{e_1, \dots, e_n\}$ for V such that $\langle e_i, e_j \rangle = 0$ and $e_i \otimes e_i = \lambda_i$. We will henceforth denote $e_i \otimes e_i$ as e_i^2 in the Clifford algebra.

If $x^2 + 1 \in K[x]$ is irreducible (e.g. $K = \mathbb{R}$), then there are (up to isomorphism) three possibilities for the values of the λ_i : $\lambda_i \in \{-1, 0, 1\}$. For example, let $K = \mathbb{R}$ and say $\lambda_i > 0$. Then $e_i^2 = \lambda_i$ and so $(e_i/\sqrt{\lambda_i})^2 = 1$. So we may replace e_i in our basis with $e_i/\sqrt{\lambda_i}$. Thus without loss of generality we may take $\lambda_i = 1$.

If $\lambda_i < 0$, then $(e_i/\sqrt{-\lambda_i})^2 = -1$, and so WLOG we may take $\lambda_i = -1$.

Over $K = \mathbb{C}$, there are only two possibilities up to isomorphism: $\lambda_i \in \{0, 1\}$. If $\lambda_i \neq 0$, then we will always be able to divide by $\sqrt{\lambda_i} \in \mathbb{C}^\times$.

Thus, without loss of generality, we may take the basis of our Clifford algebra to be an orthonormal basis

$$B = \{u_1, \dots, u_p, v_1, \dots, v_q, z_1, \dots, z_r : p + q + r = n, u_i^2 = 1, v_i^2 = -1, z_i^2 = 0\}.$$

(We may assume all norms are 1 by redefining a new inner product by scaling if we must.) So in general we have

$$\left[\left(\sum_{i=1}^p a_i u_i \right) + \left(\sum_{i=1}^q b_i v_i \right) + \left(\sum_{i=1}^r c_i z_i \right) \right]^2 = \left(\sum_{i=1}^p a_i^2 \right) - \left(\sum_{i=1}^q b_i^2 \right)$$

We will denote such a Clifford algebra as $\text{Cl}_{p,q,r}(K)$, or if $r = 0$ simply as $\text{Cl}_{p,q}(K)$.

Note that for $b \neq b' \in B$, $\langle b, b' \rangle = 0$ is not to be confused with $bb' = 0$ in the Clifford algebra! In fact, there is not much we can say about bb' as it becomes it a linearly independent generator of $\text{Cl}(V, Q)$. The most we can say is

$$bb' + b'b = (b + b')^2 - b^2 - b'^2 = Q(b + b') - Q(b) - Q(b') = \lambda_b + \lambda_{b'} - \lambda_b - \lambda_{b'} = 0,$$

so $bb' = -b'b$. Similarly, $vu = -uv$ for all $v, u \in \text{Cl}(V, Q)$.

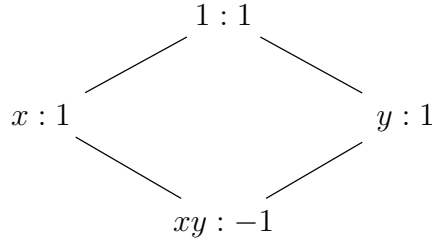
Now let us explore some examples of Clifford algebras. We will start over $K = \mathbb{R}$.

Example 1. First let us consider $\text{Cl}_{1,0}(\mathbb{R})$. Here $V = \langle x \rangle$ and so we just have $x^2 = 1$. This is identical to $\mathbb{R}[x]/(x^2 - 1)$.

Example 2. Now let us consider $\text{Cl}_{0,1}(\mathbb{R})$. Here $V = \langle x \rangle$ and $x^2 = -1$. This is identical to $\mathbb{R}[x]/(x^2 + 1) \cong \mathbb{C}$.

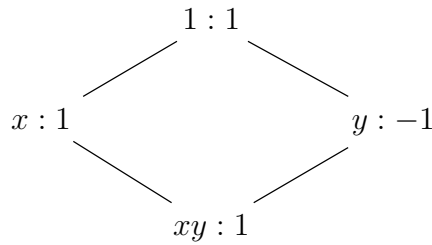
Example 3. Now let us consider $\text{Cl}_{0,0,1}(\mathbb{R})$. Here $V = \langle x \rangle$ and $x^2 = 0$. This is isomorphic to $\mathbb{R}[x]/(x^2)$.

Example 4. Now let us consider $\text{Cl}_{2,0}(\mathbb{R})$. Here $V = \langle x, y \rangle$ and $\text{Cl}_{2,0}(\mathbb{R}) = \langle 1, x, y, xy \rangle$ with the following lattice of squares.



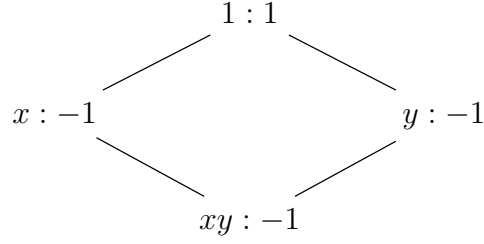
Note that $(xy)^2 = xyxy = -xxyy = -(1)(1) = -1$.

Example 5. Now let us consider $\text{Cl}_{1,1}(\mathbb{R})$. Then we have the following lattice of squares.



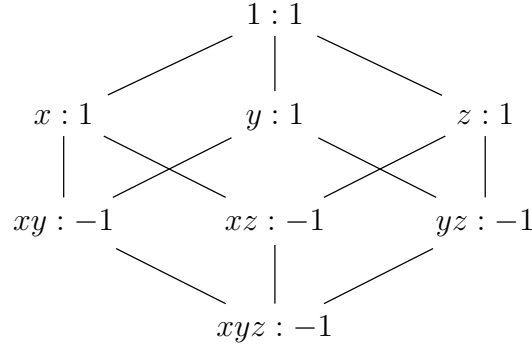
Note then that $\text{Cl}_{0,2}(\mathbb{R}) \cong \text{Cl}_{1,1}(\mathbb{R})$ under the map $xy \longleftrightarrow y$.

Example 6. Now let us consider $\text{Cl}_{0,2}(\mathbb{R})$. Then we have the following lattice of squares.



Note that $(xy)^2 = xyxy = -xxyy = -(-1)(-1) = -1$. This is isomorphic to the quaternions $\mathbb{H}$.

Example 7. Now let us consider $\text{Cl}_{3,0}(\mathbb{R})$. Then we have the following lattice of squares.

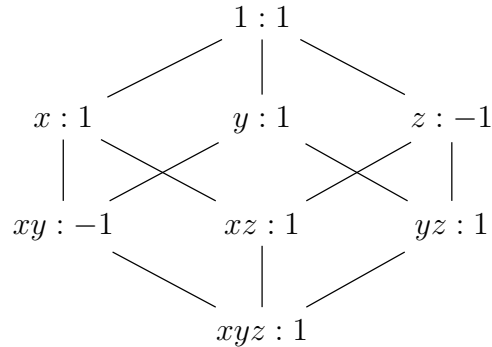


Note that $xyz = -xzy = zxy = -zyx$. In general we may compute

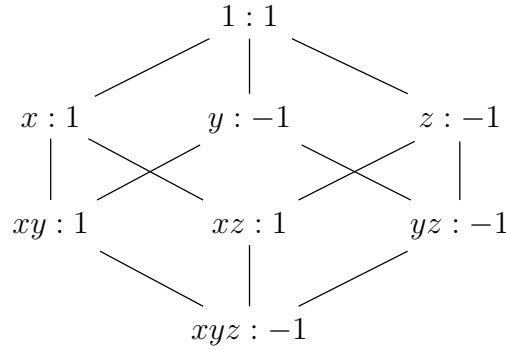
$$\prod_{j=1}^k e_{i_j} = \left(\prod_{i=1}^k e_i \right) \det(P)$$

where P is the permutation matrix corresponding to the ordered set $(e_{i_1}, \dots, e_{i_k})$.

Example 8. Now let us consider $\text{Cl}_{2,1}(\mathbb{R})$. Then we have the following lattice of squares.

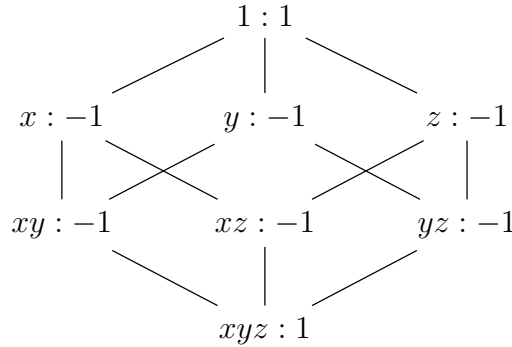


Example 9. Now let us consider $\text{Cl}_{1,2}(\mathbb{R})$. Then we have the following lattice of squares.



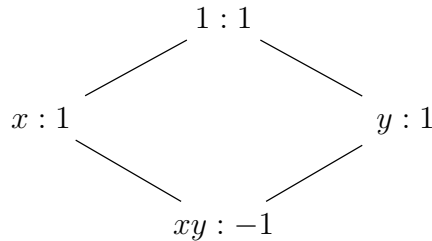
So $\text{Cl}_{1,2}(\mathbb{R}) \cong \text{Cl}_{3,0}(\mathbb{R})$.

Example 10. Now let us consider $\text{Cl}_{0,3}(\mathbb{R})$. Then we have the following lattice of squares.



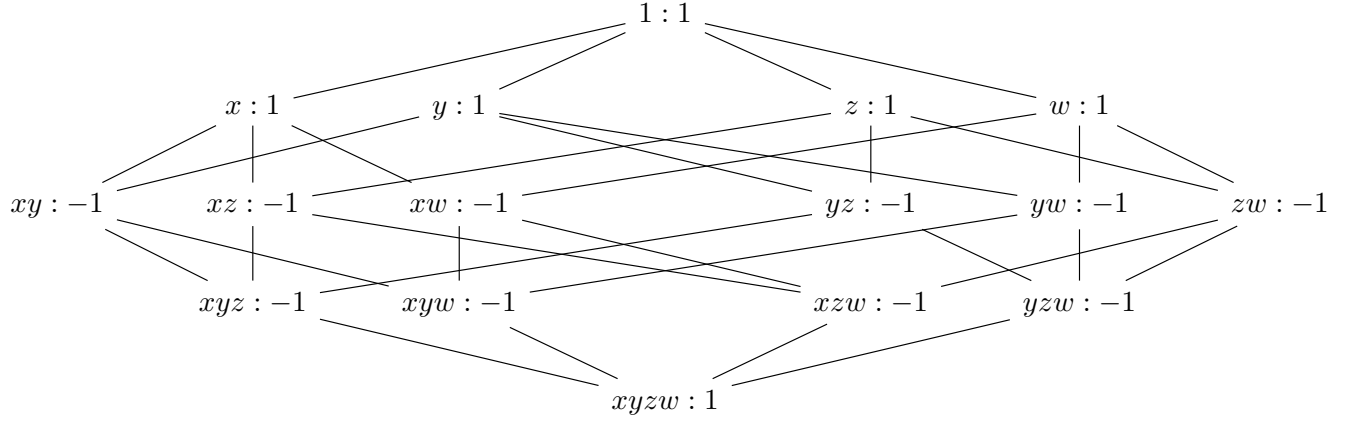
Example 11. Now let us consider $\text{Cl}_1(\mathbb{C})$. Then $V = \langle x \rangle$ and so $x^2 = 1$. Then we have $\mathbb{C}[x]/(x^2 - 1)$.

Example 12. Now let us consider $\text{Cl}_2(\mathbb{C})$. Then $V = \langle x, y \rangle$ and so $x^2 = 1$. Then we have $\mathbb{C}[x]/(x^2 - 1)$ and $\text{Cl}_2(\mathbb{C}) = \langle 1, x, y, xy \rangle$ with the following lattice of squares.

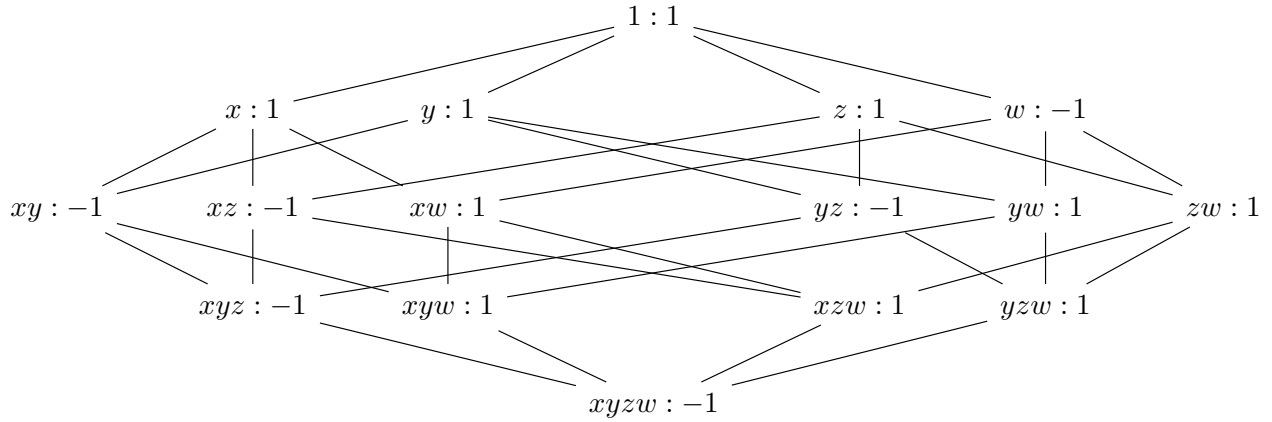


Basically, $\text{Cl}_p(\mathbb{C})$ is like $\text{Cl}_{p,0}(\mathbb{R})$ except with i there is an additional generator whose square is -1 , so it's like $\text{Cl}_{p,1}(\mathbb{R})$. Except i commutes with everything, so it's not quite like that either.

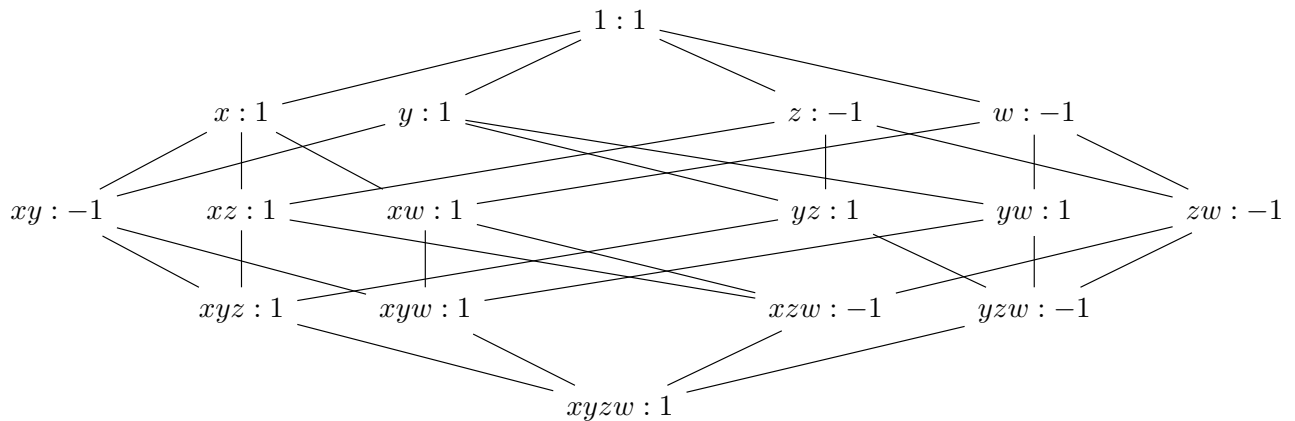
Example 13. Now let us consider $\text{Cl}_{4,0}(\mathbb{R})$. Then we have the following lattice of squares.



Example 14. Now let us consider $\text{Cl}_{3,1}(\mathbb{R})$. Then we have the following lattice of squares.

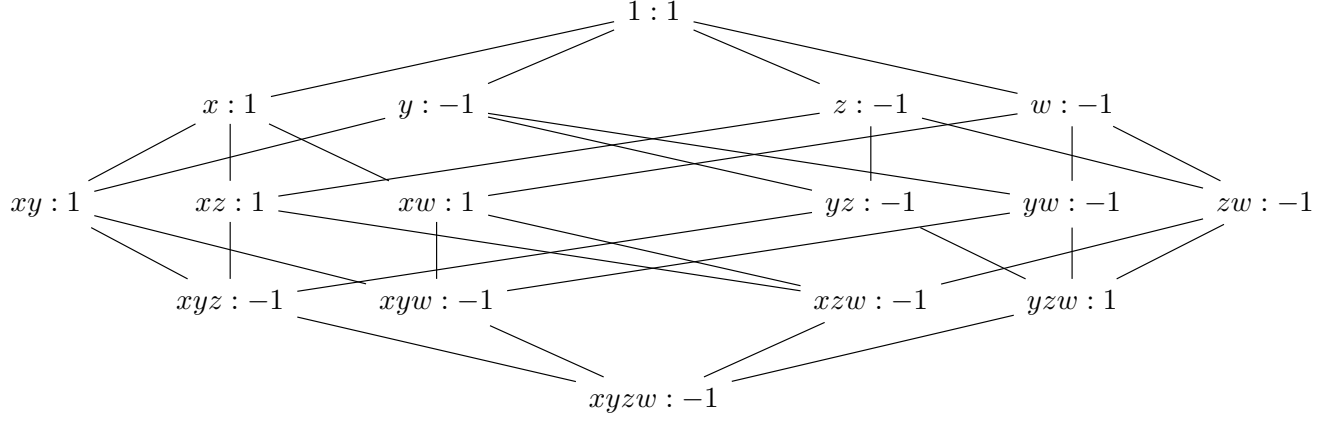


Example 15. Now let us consider $\text{Cl}_{2,2}(\mathbb{R})$. Then we have the following lattice of squares.



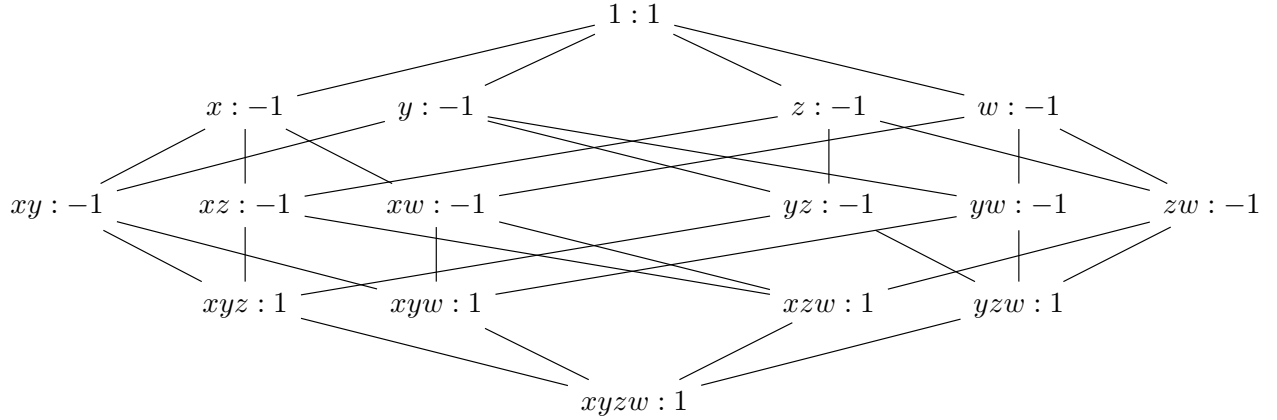
So $\text{Cl}_{2,2}(\mathbb{R}) \cong \text{Cl}_{3,1}(\mathbb{R})$.

Example 16. Now let us consider $\text{Cl}_{1,3}(\mathbb{R})$. Then we have the following lattice of squares.



So $\text{Cl}_{1,3}(\mathbb{R}) \cong \text{Cl}_{4,0}(\mathbb{R})$.

Example 17. Now let us consider $\text{Cl}_{0,4}(\mathbb{R})$. Then we have the following lattice of squares.



So $\text{Cl}_{0,4}(\mathbb{R}) \cong \text{Cl}_{1,3}(\mathbb{R}) \cong \text{Cl}_{4,0}(\mathbb{R})$.

Note that in general for $\dim V = n$, we have $\dim \text{Cl}(V, Q) = 2^n$.

Definition 3. Given a Clifford algebra $\text{Cl}(V, Q)$, the **pin group** $\text{Pin}(V, Q)$ is the group of elements of $\text{Cl}(V, Q)$ under multiplication comprising products of the form $\prod_{i=1}^k v_i$ for any $k \in \mathbb{N}$, where $Q(v_i) = \pm 1$. The **spin group** $\text{Spin}(V, Q)$ requires k be even.

Given an element $v \in \text{Cl}(V, Q)$ with $Q(v) \neq 0$, the map $w \mapsto -v w v^{-1}$ reflects w across the hyperplane perpendicular to v . So the pin group can be seen as comprising all reflection maps (of which the even elements are rotations).

For example, let $C = \text{Cl}_{3,0}$. Then

$$(x + y + z)^2 = x^2 + y^2 + z^2 = 3,$$

and so $\frac{\sqrt{3}}{3}(x + y + z) \in \text{Pin}(C)$.

By contrast, let $C = \text{Cl}_{2,1}$. Then

$$(x + y + z)^2 = x^2 + y^2 + z^2 = 1,$$

so $x + y + z \in \text{Pin}(C)$.

Let's look at some special matrix groups.

- $\text{GL}_n(K)$: **general linear group**: all invertible $n \times n$ matrices.
- $\text{SL}_n(K)$: **special linear group**: all matrices $\det M = 1$.
- $\text{PGL}_n(K)$: **projective general linear group**: all invertible $n \times n$ matrices, modded out by kI_n .
- $\text{PSL}_n(K)$: **projective special linear group**: all matrices $\det M = 1$ modded out by $-I_n$ (only matters when n is even, so $\det(-I_n) = 1$).
- $\text{O}_n(K)$: **orthogonal group**: all matrices $O^{-1} = O^T$. (Distance-preserving functions like rotations and reflections when $K = \mathbb{R}$)
- $\text{SO}_n(K)$: **special orthogonal group**: all matrices $O^{-1} = O^T$ and $\det O = 1$. (Rotations when $K = \mathbb{R}$)
- $\text{U}_n(\mathbb{C})$: **unitary group**: all matrices $U^{-1} = U^\dagger$, the conjugate transpose (Distance-preserving functions of $\mathbb{C}^n$; (Rotations of $\mathbb{C}^n$; for example, $U(1, \mathbb{C}) \cong SO(2, \mathbb{R})$)
- $\text{SU}_n(\mathbb{C})$: **special unitary group**: all matrices $U^{-1} = U^\dagger$ and $\det U = 1$
- $\text{PSU}_n(\mathbb{C})$: **special unitary group**: all matrices $U^{-1} = U^\dagger$ and $\det U = 1$, modded out by kI_n (for $k^n = 1$). Note that $iI_2 \in U_2$, but $iI_2 \notin \text{SU}_2$, although $iI_4 \in \text{SU}_4$.