Consider a quartic curve $4L \subseteq \mathbb{P}^2_k$ where k is algebraically closed.

Fix three points $P_0, Q_0, R_0 \in 4L$. We claim that for all $D \in \mathrm{Pic}^0(4L)$, there exist unique (up to permutation) points $P, Q, R \in 4L$ such that $D \sim (P - P_0) + (Q - Q_0) + (R - R_0)$.

Proof. First we will show uniqueness: Let $P - P_0 + Q - Q_0 + R - R_0 \sim P' - P_0 + Q' - Q_0 + R' - R_0$. Then $P + Q + R \sim P' + Q' + R'$. Consider the lines $\overline{PQ}$, $\overline{PR}$, and $\overline{QR}$. They each meet the quartic curve $4L$ at the two additional points a_1 and a_2, b_1 and b_2, and c_1 and c_2, respectively. Therefore

$$2P + 2Q + 2R + a_1 + a_2 + b_1 + b_2 + c_1 + c_2 \sim P' + Q' + R' + a_1 + a_2 + b_1 + b_2 + c_1 + c_2 + P + Q + R.$$

Therefore P', Q' and R' are on the cubic curve given by $a_1 + a_2 + b_1 + b_2 + c_1 + c_2 + P + Q + R$. This curve is the union of three lines, and so we must have $\{P', Q', R'\} = \{P, Q, R\}$. Thus $P + Q + R = P' + Q' + R'$ and so $P - P_0 + Q - Q_0 + R - R_0 = P' - P_0 + Q' - Q_0 + R' - R_0$.

Now consider the case $D = (P' - P'_0) + (Q' - Q'_0) + (R' - R'_0)$. We will construct the points P, Q, R such that

$$(P' - P'_0) + (Q' - Q'_0) + (R' - R'_0) \sim (P - P_0) + (Q - Q_0) + (R - R_0).$$

First note that $(P' - P'_0) + (Q' - Q'_0) + (R' - R'_0) \sim (P - P_0) + (Q - Q_0) + (R - R_0)$ if and only if

$$P' + Q' + R' + P_0 + Q_0 + R_0 \sim P + Q + R + P'_0 + Q'_0 + R'_0.$$

Consider the lines $\overline{P_0P'}$, $\overline{Q_0Q'}$ and $\overline{R_0R'}$. They each meet the quartic curve $4L$ at the two additional points p_1 and p_2, q_1 and q_2, and r_1 and r_2, respectively. Together, the nine points p_1, p_2, q_1, q_2, r_1, r_2, P'_0, Q'_0, and R'_0 determine a cubic curve $3L$ (not necessarily unique in the case of unassigned base points). The cubic $3L$ intersects $4L$ at 12 points total using Bézout's Lemma. The three additional points where $3L$ meets $4L$ are P, Q, and R. Then we have

$$P' + Q' + R' + P_0 + Q_0 + R_0 + p_1 + p_2 + q_1 + q_2 + r_1 + r_2 \sim P + Q + R + P'_0 + Q'_0 + R'_0 + p_1 + p_2 + q_1 + q_2 + r_1 + r_2$$

and result follows. By uniqueness proven earlier, this is the only choice for $\{P, Q, R\}$.

Now we wish to show that for all $P_1, P_2, Q_1, Q_2, R_1, R_2 \in 4L$, there exist $P, Q, R \in 4L$ such that

$$P_1 + P_2 - 2P_0 + Q_1 + Q_2 - 2Q_0 + R_1 + R_2 - 2R_0 \sim P - P_0 + Q - Q_0 + R - R_0.$$

Let us rewrite this equivalence as

$$P_1 + P_2 + Q_1 + Q_2 + R_1 + R_2 \sim P + P_0 + Q + Q_0 + R + R_0.$$

Let us consider the lines $\overline{P_1P_2}$, $\overline{Q_1Q_2}$, and $\overline{R_1R_2}$. They each meet $4L$ at the two additional points P_3 and P_4, Q_3 and Q_4, and R_3 and R_4, respectively. The nine points P_3, P_4, Q_3, Q_4, R_3, R_4, P_0, Q_0, and R_0 lie on a cubic curve $3L$ which intersects $4L$ at three additional points. We will call these three points P, Q, and R, and they satisfy the desired property.

Now let $D \in \text{Pic}^0(4L)$ be arbitrary. Since D has a degree of zero, we know that $D = \sum_{i=1}^{n}(A_i - B_i)$ for $n \in \mathbb{N}$ and $A_i, B_i \in 4L$ for all i. We have already shown that there exist $P_i, Q_i, R_i \in 4L$ such that

$$A_{3i-2} - B_{3i-2} + A_{3i-1} - B_{3i-1} + A_{3i} - B_{3i} \sim P_i - P_0 + Q_i - Q_0 + R_i - R_0$$

and that for each $i, j \in \mathbb{N}$ there exist $P_k, Q_k, R_k \in 4L$ such that

$$(P_i - P_0 + Q_i - Q_0 + R_i - R_0) + (P_j - P_0 + Q_j - Q_0 + R_j - R_0) \sim P_k - P_0 + Q_k - Q_0 + R_k - R_0.$$

Now if $n = 2 \mod 3$, simply add $A_{n+1} - B_{n+1} = Q_0 - Q_0$ and if $n = 1 \mod 3$, add an additional $A_{n+2} - B_{n+1} = R_0 - R_0$ to achieve desired result. □

Recall that picking 13 points on $\mathbb{P}^2$ will yield 3 unassigned base points. By adding thirteen divisors of the form $P_i - P_0$, $Q_j - Q_0$ and $R_k - R_0$, we will get a sum $P - P_0 + Q - Q_0 + R - R_0$. Then I would like to show the three unassigned base points will be the points $A, B, C \in 4L$ satisfying $A + P - 2P_0 + B + Q - 2Q_0 + C + R - 2R_0 \sim 0$ (in other words, the additive inverse of the sum of the first thirteen points).

This happens when A, B, C, P, Q, R lie on the tangent lines of P_0, Q_0, R_0 or the six points lie on the triangle of lines $\overline{P_0Q_0}$, $\overline{Q_0R_0}$, $\overline{P_0R_0}$. No, that's wrong.

This happens when the cubic given by the six points on the lines $\overline{AP}$, $\overline{BQ}$, and $\overline{CR}$ and the three points P_0, Q_0, and R_0 are tangent to $4L$ at P_0, Q_0, and R_0.

Can we pick P_0, Q_0, and R_0 such that there is a cubic that hits $4L$ only at those three points? In other words, there is a cubic $3L$ such that $3L.4L = 4P_0 + 4Q_0 + 4R_0$? Yes! One example is choosing P_0, Q_0, and R_0 to each be flex bitangents. In that case $3L$ will be the union of the three tangent lines. Another way is to choose one to be a flex bitangent and the other two to be bitangents sharing a tangent. In that case, $3L$ will be a double bitangent line and the remaining flex bitangent line.

If we pick P_0, Q_0, R_0 to be three flex bitangents, then the cubic determined by the six points on the lines $\overline{AP}$, $\overline{BQ}$, and $\overline{CR}$ and the points P_0, Q_0, and R_0 is the union of the three flex bitangent lines. This means that two of the "six" points must be P_0, two must be Q_0, and two must be R_0. This means that $\overline{AP}$ must be one of the lines $\overline{P_0Q_0}$, $\overline{Q_0R_0}$, or $\overline{Q_0R_0}$ and $\overline{BQ}$ and $\overline{CR}$ must each be one of the remaining two lines.

Thus if we choose P_0, Q_0, R_0 in this special way, $[A+B+C] = -[P+Q+R]$ if (without loss of generality) A is collinear with P_0PQ_0, ... wait... that can't be right.

$\sum_{i=1}^{4m}(P_i + Q_i + R_i) - 4mP_0 - 4mQ_0 - 4mR_0 \sim 0$ if and only if there is a degree $3m$ curve such that intersects $4L$ at all P_i, Q_i, R_i. (P_0, Q_0, R_0 flex bitangent).

For 13 points, does it matter if they are paired with P_0, Q_0, or R_0? I have to imagine yes... but... I question.

Does $P_1 + \cdots + P_{13} - 13P_0 \sim P_1 + \cdots + P_{13} - 13Q_0$? Do we have $13P_0 \sim 13Q_0$ for any $P_0, Q_0 \in 4L$? If so, then for any three points $P, Q, R \in 4L$ such that $13P_0 + P + Q + R \sim 13Q_0 + P + Q + R$. In other words, there is a quartic curve that intersects $4L$ with multiplicity 13 at P_0 and...

Say $P_1 + \cdots + P_{13} - 13P_0 \sim P - P_0 + Q - Q_0 + R - R_0$. Then $P_1 + \cdots + P_{13} + Q_0 + R_0 \sim P + Q + R + 12P_0$. Then for all $S \in 4L$, $P_1 + \cdots + P_{13} + Q_0 + R_0 + S \sim P + Q + R + 12P_0 + S$. There isn't necessarily another quartic that goes through either set of 16 points... so what next?

Let us choose P_0, Q_0, R_0 to be flex bitangent points. Given any four points $S_1, \ldots, S_4 \in 4L$, we know that

$$S_1 + S_2 + S_3 + S_4 - 4P_0 \sim S_1 + S_2 + S_3 + S_4 - 4Q_0 \sim S_1 + S_2 + S_3 + S_4 - 4R_0$$

because $4P_0 \sim 4Q_0 \sim 4R_0$. Therefore, for any 12 points $S_1, \ldots, S_{12} \in 4L$, we have

$$\sum_{i=1}^{4}(S_{3i-2} - P_0 + S_{3i-1} - Q_0 + S_{3i} - R_0) \sim \left(\sum_{i=1}^{12} S_i\right) - 12P_0.$$

Then let S_{13} be a thirteenth point. We have $\left(\sum_{i=1}^{13} S_i\right) - 13P_0 \sim T_1 - P_0 + T_2 - Q_0 - T_3 - R_0$. Define $U_1 - P_0 + U_2 - Q_0 + U_3 - R_0$ to satisfy $U_1 + T_1 - 2P_0 + U_2 + T_2 - 2Q_0 + U_3 + T_3 - 2R_0 \sim 0$. Then we claim that U_1, U_2, and U_3 are the three unassigned base points induced by the thirteen points $S_1, \ldots, S_{13}$.

$$S_1 + \cdots + S_{13} + U_1 + U_2 + U_3 \sim 14P_0 + Q_0 + R_0 \sim 6P_0 + 5Q_0 + 5R_0 \sim 3L + 2P_0 + Q_0 + R_0 \not\sim 4L?$$

Can we set $P_0 = Q_0 = R_0$? Is that legal? If so, we have the form $P + Q + R - 3P_0$ for every element of $\mathrm{Pic}^0(4L)$.