I have learned three definitions of the Weil Pairing so far:

1. From Dolgachev's *Classical Algebraic Geometry: A Modern View*, the beginning of Chapter 5: you take two divisors $\varepsilon, \varepsilon' \in E[n]$ and take $D, D' \in \mathrm{Pic}^0 E$ representing ε and ε' with disjoint supports, so $nD \sim nD' \sim 0$. Have $\mathrm{div}(f) = nD$ and $\mathrm{div}(f') = nD'$. Then the *Weil pairing* $(\varepsilon, \varepsilon') = f(D')/f'(D)$, where $g(\sum p_i) := \prod g(p_i)$.

2. I've also learned from Aftuck's masters thesis that

$$(\varepsilon, \varepsilon') = \left(\frac{f_P(Q \oplus S)}{f_P(S)}\right) \Big/ \left(\frac{f_Q(P \ominus S)}{f_Q(\ominus S)}\right)$$

where $\mathrm{div}(f_P) = nP - nO$ and $\mathrm{div}(f_Q) = nQ - nO$ and $S \in E \setminus \{O, P, \ominus Q, P \ominus Q\}$.

3. I've also learned from Wikipedia that

$$\mathrm{div}(F) = \sum_{0 \le i < n} [P \oplus k \odot Q] - \sum_{0 \le i < n} [k \odot Q]$$

and G is the translation of F by Q. Then $\mathrm{div}(G) = \mathrm{div}(F)$, so G/F is constant. Then $(\varepsilon, \varepsilon') = G/F$.

Maybe prove these are equivalent?

In part 2: show that the choice of S does not matter. For $S, S' \in E \setminus \{O, P, \ominus Q, P \ominus Q\}$, we have

$$\left(\frac{f_P(Q \oplus S)}{f_P(S)}\right) \Big/ \left(\frac{f_Q(P \ominus S)}{f_Q(\ominus S)}\right) = \left(\frac{f_P(Q \oplus S')}{f_P(S')}\right) \Big/ \left(\frac{f_Q(P \ominus S')}{f_Q(\ominus S')}\right)$$

Consider the map $F : E \to k$ defined by

$$F(S) = \left(\frac{f_P(Q \oplus S)}{f_P(S)}\right) \Big/ \left(\frac{f_Q(P \ominus S)}{f_Q(\ominus S)}\right).$$

We will show that F has no zeroes or poles: i.e., that F is constant. Note that $f_P(Q \oplus S) = 0$ if and only if $Q \oplus S = P$ and $f_P(Q \oplus S) = \infty$ if and only if $Q \oplus S = O$. In the former case, $S = P \ominus Q$ and in the latter case, $S = \ominus Q$. We will show that $\mathrm{ord}_F(S) = 0$ for all $S \in E$. If $S = \ominus Q$, then $F(S) = (f_P(O)/f_P(\ominus Q))/(f_Q(P \oplus Q)/f_Q(Q)) = (\infty^n/f_P(\ominus Q))/(f_Q(P \oplus Q)/0^n) = \infty^n 0^n/\texttt{unit}$, which results in a removable discontinuity. So $\mathrm{ord}_F(\ominus Q) = 0$.

The same goes for all points $S \in E$. Therefore $\mathrm{ord}_F(S) = 0$ for all $S \in E$ and so F is constant.

Now we will show that 1. is equivalent to 2. We want to show that for $P - O \in \mathrm{Pic}^0 E$ and $Q - O \in \mathrm{Pic}^0 E$, that $f_P(Q' - O')/f_{Q'}(P - O) = \left(\frac{f_P(Q \oplus S)}{f_P(S)}\right) \Big/ \left(\frac{f_Q(P \ominus S)}{f_Q(\ominus S)}\right)$ where $Q' - O' \sim Q - O$.

Note $f_P(Q' - O')$ in this case means $f_P(Q')/f_P(O')$. So we have $(f_P(Q')/f_P(O'))/(f_{Q'}(P)/f_{Q'}(O))$. Choosing $Q' = Q \oplus S$ under addition based at O should give us equality. We want to show that $S = O'$ in this case. Note that $Q + O' - 2O \sim Q' - O'$ because $Q + 2O' \sim Q' + 2O$??? We know that $Q' + O \sim Q + O'$. So $Q + O' + O' \sim Q' + O + O'$. No.

We have $Q' - O \sim Q + S - 2O$, so $Q' + O \sim Q + S$.

Since $S = Q' -_O Q$, we have $S - O \sim Q' - Q$ and since $Q' - O' \sim Q - O$ we have $Q' - Q \sim O' - O$. Thus $S - O \sim O' - O$ and so $S \sim O'$. Thus $S = O'$.

Now we will show that $f_{Q'}(P)/f_{Q'}(O) = f_Q(P \ominus S)/f_Q(\ominus S)$. Define $F(A) = f_{Q'}(A)/f_Q(A \ominus S)$. We will show that $\mathrm{ord}_F(A) = 0$ for all $A \in E$, and therefore that F is constant.

The potential problems are when $A \in \{Q', O'\}$. When $A = Q'$, $\mathrm{ord}_{f_{Q'}}(A) = n$ and $\mathrm{ord}_{f_Q}(A \ominus S) = \mathrm{ord}_{f_Q}(Q) = n$, so $\mathrm{ord}_F(A) = n - n = 0$. When $A = O'$, $\mathrm{ord}_{f_{Q'}}(A) = -n$ and $\mathrm{ord}_{f_Q}(A \ominus S) = -n$, and so $\mathrm{ord}_F(A) = 0$. Thus $\mathrm{ord}_F(A) = 0$ for all $A \in E$, and so $F(A) = c \in k$.

Therefore $(f_{Q'}(P)/f_{Q'}(O))/(f_Q(P \ominus S)/f_Q(\ominus S)) = F(P)/F(O) = c/c = 1$. Thus $f_{Q'}(P)/f_{Q'}(O) = f_Q(P \ominus S)/f_Q(\ominus S)$. We get our conclusion that $f_P(Q' - O')/f_{Q'}(P - O) = \left(\dfrac{f_P(Q \oplus S)}{f_P(S)}\right) \Big/ \left(\dfrac{f_Q(P \ominus S)}{f_Q(\ominus S)}\right)$ and so definitions 1. and 2. are equivalent.