Let $X = \{(t^3 : t^2u : tu^2 : u^3) : (t : u) \in \mathbb{P}^1\} \subseteq \mathbb{P}^3$ be a twisted cubic. Then $X \subseteq V(xw - yz, y^2 - xz)$; specifically, $X + V(x, y) = V(xw - yz, y^2 - xz)$. We can thus know that $\deg X = 3$ because two quadric surfaces must intersect at a degree-4 curve, but one of the components of $Q_1 \cap Q_2$ is a line, which leaves X as degree three. You can see it here.

Then the tangent variety $\mathcal{T}_X$ is the image of the tangent lines in the Grassmannian $\mathfrak{Gr}(2, 4)$, and the secant variety $\mathcal{S}_X$ is the closure of the image of the secant lines in $\mathfrak{Gr}(2, 4)$.

Then the dimension of $\mathcal{T}_X$ is 1 and the degree is 4. We can see this the following way: note that X is a rational curve (parametrized by $\mathbb{P}^1$) and so $\mathcal{T}_X$ can also be parametrized by $\mathbb{P}^1$ and so is not only a curve in $\mathfrak{Gr}(2, 4)$, but a rational curve as well.

To find the degree of $\mathcal{T}_X$, we want to count how many times a general three-dimensional hyperplane in $\mathfrak{Gr}(2, 4)$ intersects $\mathcal{T}_X$. All three-dimensional hyperplanes in $\mathfrak{Gr}(2, 4)$ are rationally equivalent to Σ_1, the Schubert class of all lines touching some given line. Thus we can take general line $L = \overline{(a_0 : a_1 : a_2 : a_3)(b_0 : b_1 : b_2 : b_3)}$ in $\mathbb{P}^3$ and count how many tangent lines of X touch L.

Take the point $[t] = (t^3, t^2, t) \in \mathfrak{D}(w) \cong \mathbb{A}^3$. Then the tangent line to $X|_{\mathfrak{D}(w)}$ at (t^3, t^2, t) intersects $V(w)$ at $(3t^2 : 2t : 1 : 0)$. So we can form the tangent line $T_{[t]}(X) = \overline{(t^3 : t^2 : t : 1)(3t^2 : 2t : 1 : 0)}$. Then $T_{[t]}(X) \cap L \neq \varnothing$ if and only if

$$\begin{vmatrix} t^3 & t^2 & t & 1 \\ 3t^2 & 2t & 1 & 0 \\ a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 \end{vmatrix} = 0.$$

This is a degree-4 polynomial, so there must be four values for t whose tangent line $T_{[t]}(X)$ meets L. Thus $\mathcal{T}_X$ must be degree 4.

In other words, $\mathcal{T}_X.\sigma_1 = 4\sigma_{2,2}$. We know that $\sigma_1.\sigma_{2,1} = \sigma_{2,2}$, so $\mathcal{T}_X = 4\sigma_{2,1}$.

Note that this method only worked because we could parametrize X over $\mathbb{P}^1$. In general I believe this is much harder. For example, we could define $Q_1 = V(xw - yz)$ and $Q_2 = V(y^2 - xz)$ (note Q_2 is singular at $(0 : 0 : 0 : 1)$). Then for each $P = (p_0 : p_1 : p_2 : p_3) \in \mathbb{P}^3$, define $T_P(Q_1) = p_3x - p_2y - p_1z + p_0w$ and $T_P(Q_2) = p_2x - 2p_1y + p_0z$. Then we can realize our general line L as the intersection of planes $V(\pi_0x + \pi_1y + \pi_2z + \pi_3w, \pi'_0x + \pi'_1y + \pi'_2z + \pi'_3w)$. Then $T_P(X) = T_P(Q_1) \cap T_P(Q_2)$ and $T_P(X) \cap L \neq 0$ only if

$$\begin{vmatrix} p_3 & -p_2 & -p_1 & p_0 \\ p_2 & -2p_1 & p_0 & 0 \\ \pi_0 & \pi_1 & \pi_2 & \pi_3 \\ \pi'_0 & \pi'_1 & \pi'_2 & \pi'_3 \end{vmatrix} = 0.$$

This is a degree-2 surface with variables $p_0, \ldots, p_3$, which intersects $Q_1 \cap Q_2$ at eight points, two of which are at the line $V(x, y)$ and can be disregarded immediately, but that still leaves six points on X instead of four as we counted previously. Why the discrepancy? It's because we are overcounting points that are singular on Q_2, and also this method would overcount planes that coincide on both Q_1 and Q_2 in the case both are smooth. We will call both scenarios collectively "singular planes." To fix this, we would have to count the overcount, which seems difficult.

Maybe we can use Jacobi's formula to simplify the search for singular tangent planes:

$$\frac{\partial \det(A(x_0,\ldots,x_n))}{\partial x_i} = \operatorname{tr}\left(\operatorname{adj}(A)\frac{\partial A(x_0,\ldots,x_n)}{\partial x_i}\right).$$

Or maybe we can just go to the secant variety $\mathcal{S}_X$, with $\mathcal{S}_X^\circ = \bigcup_{x\neq x'\in X}\overline{xx'}$, and $\mathcal{S}_X = \overline{\mathcal{S}_X^\circ}$. Particularly, $\mathcal{S}_X^\circ = \mathcal{S}_X \setminus \mathcal{T}_X$. First note that $\mathcal{S}_X$ is two-dimensional. We can see this by noting that $(X\times X)\setminus\Delta \twoheadrightarrow \mathcal{S}_X^\circ$ is a degree-2 map where the images of (x,x') and (x',x) are both $\overline{xx'}$. We know that this map is only 2-to-1 because every secant line of X is also a secant line of Q_1 and Q_2, which are degree 2, and so must intersect only twice.

Next let us find the degree of $\mathcal{S}_X \subseteq \mathfrak{Gr}(2,4)$. We wish to know how many times $\mathcal{S}_X$ intersects a general plane in $\mathfrak{Gr}(2,4)$. The planes in $\mathfrak{Gr}(2,4)$ come in two rational equivalence classes: $\sigma_{1,1}$ (the class of lines contained in a given plane) and σ_2 (the class of lines containing a given point).

We can see that $\mathcal{S}_X.\sigma_{1,1} = 3\sigma_{2,2}$ because for each plane $\Pi\subseteq\mathbb{P}^3$, Π intersect X at three points: these must be three non-collinear points because otherwise the line through the three collinear points would intersect Q_1 and Q_2 at three points, contradicting Bézout's since they are quadric.

Thus of all the lines on Π, three of them are secant to X: one for every pair of the three noncollinear points.

Next let us find $\mathcal{S}_X.\sigma_2$. Maybe we can use undetermined coefficients to say $\mathcal{S}_X = 3\sigma_{1,1} + b\sigma_2$. Then $\mathcal{S}_X.\sigma_1 = (3+b)\sigma_{2,1}$. Maybe we can use the fact that $\mathcal{T}_X = 4\sigma_{2,1}$?

We can let $P = (p_0,p_1,p_2,p_3)\in\mathbb{P}^3$ be general. Then $(t^3,t^2u,tu^2,u^3),(r^3,r^2s,rs^2,s^3)\in X$. Then the three points are collinear if and only if

$$\operatorname{rank}\begin{pmatrix} p_0 & p_1 & p_2 & p_3\\ t^3 & t^2u & tu^2 & u^3\\ r^3 & r^2s & rs^2 & s^3\end{pmatrix} = 2.$$

Let us restrict to the affine space $\mathfrak{D}(w)$ (letting $p_3=1$), then

$$\operatorname{rank}\begin{pmatrix} p_0 & p_1 & p_2 & 1\\ t^3 & t^2 & t & 1\\ r^3 & r^2 & r & 1\end{pmatrix} = 2,$$

so

$$\begin{pmatrix} r^3\\ r^2\\ r\\ 1\end{pmatrix} = c_1\begin{pmatrix} t^3\\ t^2\\ t\\ 1\end{pmatrix} + c_2\begin{pmatrix} p_0\\ p_1\\ p_2\\ 1\end{pmatrix},$$

so

$$c_1 + c_2 = 1,$$
$$(c_1t + c_2p_2)^2 = c_1t^2 + c_2p_1,$$
$$(c_1t + c_2p_2)^3 = c_1t^2 + c_2p_0.$$

Thus

$$(c_1t + (1-c_1)p_2)^2 = c_1t^2 + (1-c_1)p_1,$$

$$(c_1 t + (1 - c_1)p_2)^3 = c_1 t^2 + (1 - c_1)p_0.$$

Maybe instead we can look at Chern classes of some projective bundle. Take $P \in \mathbb{P}^3$ to be general. Then $\Sigma_1(P)$ is the set of planes containing P. Then X determines a section of the bundle of three points in $\Sigma_1(P)$. As the planes in $\Sigma_1(P)$ vary, how many times do we expect to see two points of X collinear with P?

After messing around in Macaulay2, I keep getting one secant line per point. Here's how: take the matrix in affine coordinates

$$\begin{pmatrix} p_0 & p_1 & p_2 & p_3 \\ t^3 & t^2 u & tu^2 & u^3 \\ r^3 & r^2 s & rs^2 & s^3 \end{pmatrix}$$

and find the ideal I of 3×3 minors. According to Macaulay2, I is minimally generated by a quadric and a cubic in the variables t and r. But both generators are divisible by $t - r$ since every tangent line will cause the rank of the matrix to shrink to 2. So dividing $t - r$ out, we get a line and a quadric that intersect at two points in $\mathbb{A}^2$. So there is one secant line corresponding to that pair of points.

An actual argument: project the curve X from the general point P into a plane. Then the image of the projection is still a degree-3 curve (take a line in the plane; the line extends to a plane in $\mathbb{P}^3$ by adjoining P; that plane intersects X at three points; the images of those points are the three places the line intersects the image of X). The image also has genus 3. By the genus-degree formula, the image of X must have a singularity of multiplicity 2. Thus the preimage of the singularity yields the sole secant line of X containing P.

(In general, when you project a curve from a point not on the curve, the degree remains the same, but when you project from a point on the curve, the degree drops by 1.)

Thus $\mathcal{S}_X.\sigma_2 = 1$. Thus $\mathcal{S}_X = 3\sigma_{1,1} + \sigma_2$. So $\mathcal{S}_X^2 = 9 + 1 = 10$. And $\mathcal{S}_X.\sigma_1 = 4\sigma_{2,1} = \mathcal{T}_X$. A general hyperplane in $\mathfrak{Gr}(2,4)$ is of the class σ_1 and so a general plane has the class $\sigma_1^2 = \sigma_{1,1} + \sigma_2$. Thus the degree of $\mathcal{S}_X$ is $\mathcal{S}_X.\sigma_1^2 = 3 + 1 = 4$.