1 t-Norms and Fuzzy Logic

Definition 1. A **t-norm** is a function $\mathsf{T} : [0,1] \times [0,1] \to [0,1]$ that satisfies the following four properties:

1. Commutativity,
2. Monotonicity: $\mathsf{T}(a,b) \le \mathsf{T}(c,d)$ if $a \le c$ and $b \le d$,
3. Associativity: $\mathsf{T}(a, \mathsf{T}(b,c)) = \mathsf{T}(\mathsf{T}(a,b),c)$,
4. the number 1 acts as an identity: $\mathsf{T}(a,1) = a$.

Definition 2. A **negator** $n : [0,1] \to [0,1]$ is a monotonically decreasing function where $n(0) = 1$ and $n(1) = 0$. A negator is called

- **strict** if it is strictly decreasing, and
- **strong** if it is strict and involutive: $n(n(x)) = x$ for all $x \in [0,1]$.

Definition 3. A **t-conorm** is a function $\perp: [0,1] \times [0,1] \to [0,1]$ that satisfies the following four properties:

1. Commutativity,
2. Monotonicity: $\perp(a,b) \le \perp(c,d)$ if $a \le c$ and $b \le d$,
3. Associativity: $\perp(a, \perp(b,c)) = \perp(\perp(a,b),c)$,
4. the number 0 acts as an identity: $\perp(a,0) = a$.

Given a t-norm T and a negator n, the **t-conorm** induced by (T, n) is a function $\perp: [0,1] \times [0,1] \to [0,1]$ such that $\perp(a,b) = n(\mathsf{T}(n(a), n(b)))$. If n is a strong negator, $(\mathsf{T}, \perp, n)$ is a **De Morgan triplet**.

Throughout, we will use the negator $n(x) = 1 - x$. There are a number of different interesting t-norms and their respective t-conorms, as tabulated below.

Type	t-Norm	t-Conorm
Gödel	$\mathsf{T}(a,b) = \min(a,b)$	$\perp(a,b) = \max(a,b)$
Probabilistic	$\mathsf{T}(a,b) = ab$	$\perp(a,b) = a + b - ab$
Łukasiewicz	$\mathsf{T}(a,b) = \max(0, a+b-1)$	$\perp(a,b) = \min(1, a+b)$
Drastic	$\mathsf{T}(a,b) = \begin{cases} b & a = 1 \\ a & b = 1 \\ 0 & \text{otherwise} \end{cases}$	$\perp(a,b) = \begin{cases} b & a = 0 \\ a & b = 0 \\ 1 & \text{otherwise} \end{cases}$
Nilpotent	$\mathsf{T}(a,b) = \begin{cases} \min(a,b) & a+b > 1 \\ 0 & \text{otherwise} \end{cases}$	$\perp(a,b) = \begin{cases} \max(a,b) & a+b < 1 \\ 1 & \text{otherwise} \end{cases}$
Einstein-Hamacher	$\mathsf{T}(a,b) = \begin{cases} 0 & a = b = 0 \\ \frac{ab}{a+b-ab} & \text{otherwise} \end{cases}$	$\perp(a,b) = \frac{a+b}{1+ab}$

A fuzzy logic system is one that assigns truth values (veracity) to propositions: given a proposition P, $v(P) \in [0,1]$ is its veracity. Then we use a De Morgan triplet $(\mathsf{T}, \perp, n)$ to perform algebra on the veracities of (combinations of) conjunctions, disjunctions, and negations of propositions. In particular, given propositions P and Q, we have the following:

- $v(P \wedge Q) = \mathsf{T}(v(P), v(Q))$,
- $v(P \vee Q) = \perp(v(P), v(Q))$,
- $v(\neg P) = n(v(P))$.

Given two propositions P and Q, the proposition $P \Rightarrow Q$ is taken to mean the disjunction $Q \vee \neg P$. Thus

$$v(P \Rightarrow Q) = \perp(v(Q), n(v(P))).$$

Let's look at the equation

$$v(P \Rightarrow Q) = v(Q)$$

under different norms. First, under the Gödel norms, we have

$$\max(v(Q), 1 - v(P)) = v(Q),$$

and so

$$v(Q) \geq 1 - v(P),$$

and so $v(Q) + v(P) \geq 1$.

Under the probabilisitc norms,

$$v(P) + v(Q) - v(P)v(Q) = v(Q),$$

so

$$v(P)(1 - v(Q)) = 0$$

so $v(P) = 0$ or $v(Q) = 1$.

Under Łukasiewicz norms,

$$\min(1, 1 - v(P) + v(Q)) = v(Q)$$

so either $v(Q) = 1$ or $v(Q) = 1 - v(P) + v(Q)$ and so $v(P) = 0$.

Under Einstein-Hamacher,

$$\frac{v(Q) + 1 - v(P)}{1 + (1 - v(P))v(Q)} = v(Q)$$

$$v(Q) + 1 - v(P) = v(Q) + v(Q)^2 - v(Q)^2 v(P),$$

so

$$v(Q)^2(1 - v(P)) = 1 - v(P)$$

so either $v(P) = 1$ or $v(Q) = 1$.

2 Łukasiewicz Fuzzy Logic

Łukasiewicz fuzzy logic uses the standard negation $n(x) = 1 - x$ two conjuctions (weak $\wedge$ and strong $\otimes$) and two disjunctions (weak $\vee$ and strong $\oplus$):

- $v(P \wedge Q) = \min(v(P), v(Q))$ (Gödel)
- $v(P \otimes Q) = \max(0, v(P) + v(Q) - 1)$ (Łukasiewicz)
- $v(P \vee Q) = \max(v(P), v(Q))$ (Gödel)
- $v(P \oplus Q) = \min(1, v(P) + v(Q))$ (Łukasiewicz)

and uses strong disjunction for implications and strong conjunction for logical equivalence:

- $v(P \implies Q) = v(Q \oplus \neg P) = \min(1, v(Q) + 1 - v(P))$,
- $v(P \iff Q) = v((P \implies Q) \otimes (Q \implies P)) = \max(0, \min(1, v(Q) + 1 - v(P)) + \min(1, v(P) + 1 - v(Q)) - 1) = 1 - |v(P) - v(Q)|$.

Łukasiewicz fuzzy logic also has a **doubtful** proposition: $\delta(P) = P \iff \neg P$, and a modal logic:

- $\diamond P := \neg P \implies P$,
- $\Box P := \neg \diamond \neg P$.

This gives us the following veracities:

- $v(\diamond P) = v(\neg P \implies P) = v(P \oplus \neg\neg P) = v(P \oplus P) = \min(1, 2v(P))$,
- $v(\Box P) = v(\neg \diamond \neg P) = v(\neg(P \implies \neg P)) = v(\neg(\neg P \oplus \neg P)) = v(P \otimes P) = \max(0, 2v(P) - 1)$.
- $v(\delta(P)) = v(P \iff \neg P) = 1 - |v(P) + 1 - v(P)| = 0$.

Originally, Łukasiewicz logic was built from the constant F and the operation $\implies$, with all other constants and operations following:

- $\neg P := P \implies F$
- $P \vee Q := (P \implies Q) \implies Q$
- $P \wedge Q := \neg(\neg P \vee \neg Q)$
- $P \iff Q := (P \implies Q) \wedge (Q \implies P)$ (something is wrong here; this is supposed to be strong disjunction?)
- $T := F \implies F$
- $P \oplus Q := \neg P \implies Q$
- $P \otimes Q := \neg(P \implies \neg Q)$