Consider for example the field $\mathbb{F}_4 = \mathbb{F}_2[\alpha]$ where $\alpha^2 = \alpha + 1$. Then the polynomial $x^3 + \alpha \in \mathbb{F}_4[x]$ is irreducible. Then let $\mathbb{F}_{64} = \mathbb{F}_4[\beta]$ where $\beta^3 = \alpha$.

Then we can consider the plane $[A, B]$ with $A, B \in \mathbb{F}_{64}$. We can write $A = a_0 + a_1\beta + a_2\beta^2$ and $B = b_0 + b_1\beta + b_2\beta^2$ for $a_i, b_i \in \mathbb{F}_4$.

The plane $[A, B]$ is the set $\{(x, Ax + Bx^4) : x \in \mathbb{F}_{64}\}$. That is, we can choose $x = 1, \beta, \beta^2$ to identify three points contained in the plane. For $x = 1$, we get $(1, A + B) = (1, 0, 0, a_0 + b_0, a_1 + b_1, a_2 + b_2)$. For $x = \beta$, we get

$$Ax + Bx^4 = A\beta + B\beta\alpha = a_0\beta + a_1\beta^2 + a_2\alpha + b_0\beta\alpha + b_1\beta^2\alpha + b_2\alpha + b_2.$$

So $(\beta, A\beta + B\beta^4)$ corresponds to the point $(0, 1, 0, a_2\alpha + b_2\alpha + b_2, a_0 + b_0\alpha, a_1 + b_1\alpha)$.

Finally, for $x = \beta^2$, we have $A\beta^2 + B\beta^8 = A\beta^2 + B\beta^2(\alpha + 1)$.

$$A\beta^2 + B\beta^2(\alpha + 1) = a_0\beta^2 + a_1\alpha + a_2\beta\alpha + b_0\beta^2(\alpha + 1) + b_1 + b_2\beta$$

and so $(\beta^2, A\beta^2 + B\beta^8)$ corresponds to the point $(0, 0, 1, a_1\alpha + b_1, a_2\alpha + b_2, a_0 + b_0\alpha + b_0)$. So

$$\begin{aligned}
[A, B] = \text{join}\{&(1, 0, 0, a_0 + b_0, a_1 + b_1, a_2 + b_2), \\
&(0, 1, 0, a_2\alpha + b_2\alpha + b_2, a_0 + b_0\alpha, a_1 + b_1\alpha), \\
&(0, 0, 1, a_1\alpha + b_1, a_2\alpha + b_2, a_0 + b_0\alpha + b_0)\}.
\end{aligned}$$

In particular, the plane

$$\begin{aligned}
[A, 0] = \text{join}\{&(1, 0, 0, a_0, a_1, a_2), \\
&(0, 1, 0, a_2\alpha, a_0, a_1), \\
&(0, 0, 1, a_1\alpha, a_2\alpha, a_0)\}.
\end{aligned}$$

Also define the plane $\Pi_\infty = \text{join}\{(0, 0, 0, 1, 0, 0), (0, 0, 0, 0, 1, 0), (0, 0, 0, 0, 0, 1)\}$.

I claim that $\mathcal{S} = \{[A, 0] : A \in \mathbb{F}_{64}\} \cup \{\Pi_\infty\}$ is a spread of $\mathbb{P}^5_{\mathbb{F}_4}$. First note that $\#\mathbb{P}^5_{\mathbb{F}_4} = \dfrac{4^6 - 1}{4 - 1} = 1365$. And $\#\mathbb{P}^2_{\mathbb{F}_4} = 21$. And $1365/21 = \#\mathcal{S}$.

$$\begin{vmatrix}
1 & 0 & 0 & a_0 & a_1 & a_2 \\
0 & 1 & 0 & \alpha a_2 & a_0 & a_1 \\
0 & 0 & 1 & \alpha a_1 & \alpha a_2 & a_0 \\
1 & 0 & 0 & b_0 & b_1 & b_2 \\
0 & 1 & 0 & \alpha b_2 & b_0 & b_1 \\
0 & 0 & 1 & \alpha b_1 & \alpha b_2 & b_0
\end{vmatrix} = (a_0 + b_0)^3 + \alpha(a_1 + b_1)^3 + \alpha^2(a_2 + b_2)^3 + \alpha(a_0 + b_0)(a_1 + b_1)(a_2 + b_2),$$

which we can rewrite as

$$X^3 + \alpha Y^3 + \alpha^2 Z^3 + \alpha XYZ \in \mathbb{F}_4[X, Y, Z].$$

We want to show the only root in $\mathbb{F}_4$ is $(X, Y, Z) = (0, 0, 0)$. Suppose $X \neq 0$. Then $X^3 = 1$. So we have $1 + \alpha Y^3 + \alpha^2 Z^3 + \alpha XYZ$.

For the case $\mathbb{F}_2$, we can construct a spread via the field extension $\mathbb{F}_8 \cong \mathbb{F}_2/(x^3 + x + 1)$ and defining the planes

$$[A, B] = \text{join}\{(1, 0, 0, a_0 + b_0, a_1 + b_1, a_2 + b_2),$$
$$(0, 1, 0, a_2 + b_1, a_0 + a_2 + b_1 + b_2, a_1 + b_0 + b_2),$$
$$(0, 0, 1, a_1 + b_1 + b_2, a_1 + a_2 + b_0 + b_1, a_0 + a_2 + b_0 + b_1 + b_2)\}$$

for $A, B \in \mathbb{F}_8$ and the spread comprising

$$[A, 0] = \text{join}\{(1, 0, 0, a_0, a_1, a_2),$$
$$(0, 1, 0, a_2, a_0 + a_2, a_1),$$
$$(0, 0, 1, a_1, a_1 + a_2, a_0 + a_2)\}.$$

Do $[0, 0]$, $[1, 0]$, and Π_∞ form a "regulus" in $\mathbb{P}^5_{\mathbb{F}_2}$? We have $[0, 0] = V(x_3, x_4, x_5)$, $[1, 0] = V(x_0 + x_3, x_1 + x_4, x_2 + x_5)$, and $\Pi_\infty = V(x_0, x_1, x_2)$. Is there some quadric 4-dimensional hypersurface containing all three planes?

The family

$$I = (x_0 x_4 + x_1 x_3, x_0 x_5 + x_2 x_3, x_1 x_5 + x_2 x_4)$$

contains all three planes. According to Macaulay2, I is codimension 2 and degree 3, so it carves out a cubic threefold containing the planes.

I should further study the plane-bearing properties of cubic threefolds in $\mathbb{P}^5_k$, for example

$$V(x_0 x_4 - x_1 x_3, x_0 x_5 - x_2 x_3, x_1 x_5 - x_2 x_4).$$

I believe this is isomorphic to the Segre embedding

$$\mathbb{P}^2 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^5$$

where

$$(a_0, a_1, a_2), (b_0, b_1) \mapsto (a_0 b_0, a_0 b_1, a_1 b_0, a_1 b_1, a_2 b_0, a_2 b_1).$$

In this case, we have

$$V(x_0 x_3 - x_1 x_2, x_0 x_5 - x_1 x_4, x_2 x_5 - x_3 x_4),$$

but this is isomorphic to the above variety via $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$.

We can look in the Grassmannian of planes in $\mathbb{P}^5$. $\mathfrak{Gr}(3, 6)$ is 9-dimensional and embeds in $\mathbb{P}^{19}$ via Plücker. According to Macaulay2, the degree of the Plücker embedding of $\mathfrak{Gr}(3, 6) \hookrightarrow \mathbb{P}^{19}$ is 42.

```
i1:  loadPackage "SchurRings"
```

```
i2:   S=schurRing(QQ,s,3)

i3:   s_1^9
```

The coefficient of the $s_{3,3,3}$ is 42. The $\mathbb{P}^2 \times \mathbb{P}^1$ is some kind of curve in $\mathfrak{Gr}(3,6)$. Is it a curve of degree 3?

Or maybe... we use the Chow rings. $A(\mathbb{P}^2) \cong \mathbb{Z}[L]/(L^3)$ and $A(\mathbb{P}^1) \cong \mathbb{Z}[P]/(P^2)$, and $A(\mathbb{P}^5) \cong \mathbb{Z}[H]/(H^6)$. So we have the function of rings

$$\mathbb{Z}[H]/(H^6) \to \mathbb{Z}[L]/(L^3) \otimes \mathbb{Z}[P]/(P^2)$$

where $A^0(\mathbb{P}^2 \times \mathbb{P}^1)$ is generated by $1 \otimes 1$, A^1 is generated by $L \otimes 1$ and $1 \otimes P$, A^2 is generated by $L^2 \otimes 1$ and $L \otimes P$, and A^3 is generated by $L^2 \otimes P$. Identifying $\ell = L \otimes 1$ and $p = 1 \otimes P$, we have $A(\mathbb{P}^2 \times \mathbb{P}^1) \cong \mathbb{Z}[\ell, p]/(\ell^3, p^2)$. Then for consistency let's write $h = H$ and we have

$$f : \mathbb{Z}[h]/(h^6) \to \mathbb{Z}[\ell, p]/(\ell^3, p^2)$$

where $h \mapsto a\ell + bp$, $h^2 \mapsto a^2\ell^2 + 2ab\ell p$, $h^3 \mapsto 3a^2b\ell^2 p$, and $h^4 \mapsto 0$. We can find a and b through a few tests: $f(h)p = a\ell p$. The class p represents a general plane $\mathbb{P}^2 \times \{*\} \subseteq \mathbb{P}^2 \times \mathbb{P}^1$. Then ℓp is the class of a line contained in a specific section $L \times \{*\}$.

Maybeee... we can construct a specific h whose intersection with the Segre variety is a $\mathbb{P}^2 \times \{*\} \cup \mathbb{P}^1 \times \mathbb{P}^1$. Yeah, take the two classes $\mathbb{P}^2 \times \{*\}, \mathbb{P}^1 \times \mathbb{P}^1 \subseteq \mathbb{P}^2 \times \mathbb{P}^1$: they meet at the line $\mathbb{P}^1 \times \{*\}$. The $\mathbb{P}^1 \times \mathbb{P}^1$ is contained in a $\mathbb{P}^3$ which must necessarily contain the line $\mathbb{P}^1 \times \{*\}$. Then it is only a matter of including one of the points of $\mathbb{P}^2 \times \{*\}$ not shared by the $\mathbb{P}^1 \times \mathbb{P}^1$, and that extra point extends the $\mathbb{P}^3$ to a $\mathbb{P}^4$, which is the class h. So in fact $f(h) = \ell + p$, and so $h^3 = 3\ell^2 p$, so the degree of the Segre embedding $\mathbb{P}^2 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^5$ is 3.

Similarly, the degree of the Segre embedding $\mathbb{P}^2 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^8$ is $(\ell_1 + \ell_2)^4 = 6\ell_1^2\ell_2^2$.

In general, the degree of the Segre embedding $\mathbb{P}^n \times \mathbb{P}^m \hookrightarrow \mathbb{P}^N$ is $\binom{n+m}{n}$?