1 Three Functions

In the popular game of chance *Rock, Paper, Scissors*, contestants simultaneously choose to cast either rock, paper, or scissors to determine a winner. The rules are:

`rock beats scissors`
`scissors beats paper`
`paper beats rock`.

We can set up a system of differential equations that reflects this balance. For example, we can set up an equation

$$r' = s - p,$$

saying that the population of rocks grows the more scissors there are (because rocks eat scissors) and shrinks the more papers there are (because the papers eat the rocks). Setting up the whole system like this gives us

$$\begin{aligned} r' &= -p + s \\ p' &= r - s \\ s' &= -r + p \end{aligned}$$

which can be turned into the matrix equation

$$\begin{bmatrix} r' \\ p' \\ s' \end{bmatrix} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} r \\ p \\ s \end{bmatrix}.$$

The eigenvalues of the matrix $\begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}$ can be found by finding the roots of the characteristic polynomial

$$\det \begin{bmatrix} -\lambda & -1 & 1 \\ 1 & -\lambda & -1 \\ -1 & 1 & -\lambda \end{bmatrix} = -\lambda(\lambda^2 + 1) - (-1)(-\lambda - 1) + (1)(1 - \lambda) = -\lambda^3 - 3\lambda$$

which factors as $-\lambda(\lambda^2 + 3)$. So the roots are $\lambda = 0$, $\lambda = i\sqrt{3}$, and $\lambda = -i\sqrt{3}$.

To find the eigenvector associated to $\lambda = 0$, plug 0 in for λ and solve

$$\begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

We get the equations $0k_1 - k_2 + k_3 = 0$, $k_1 + 0k_2 - k_3 = 0$, and $-k_1 + k_2 + 0k_3 = 0$. The first equation tells us $k_2 = k_3$, the second equation tells us $k_1 = k_3$, and the third equation is redundant: it tells us $k_1 = k_2$. A vector satisfying these equations is

$$K = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Therefore our solution will contain $Y_1 = \begin{bmatrix}1\\1\\1\end{bmatrix} e^{0x} = \begin{bmatrix}1\\1\\1\end{bmatrix}$.

Next, let's look at the eigenvalue $\lambda = i\sqrt{3}$. Plugging this eigenvalue in, we get the equation

$$\begin{bmatrix}-i\sqrt{3} & -1 & 1\\ 1 & -i\sqrt{3} & -1\\ -1 & 1 & -i\sqrt{3}\end{bmatrix}\begin{bmatrix}k_1\\k_2\\k_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}.$$

we get the equations

$$\begin{aligned}-i\sqrt{3}k_1 - k_2 + k_3 &= 0\\ k_1 - i\sqrt{3}k_2 - k_3 &= 0\\ -k_1 + k_2 - i\sqrt{3}k_3 &= 0.\end{aligned}$$

Choosing $k_1 = 1$, we get

$$\begin{aligned}-k_2 + k_3 &= i\sqrt{3}\\ -i\sqrt{3}k_2 - k_3 &= -1\\ k_2 - i\sqrt{3}k_3 &= 1.\end{aligned}$$

The first equation tells us $k_3 = i\sqrt{3} + k_2$. Plugging $i\sqrt{3} + k_2$ in for k_3 in the second equation gives us

$$-i\sqrt{3}k_2 - i\sqrt{3} - k_2 = -1.$$

Solving for k_2 yields

$$\begin{aligned}k_2(-1 - i\sqrt{3}) - i\sqrt{3} &= -1\\ k_2(-1 - i\sqrt{3}) &= -1 + i\sqrt{3}\\ k_2 &= \frac{-1 + i\sqrt{3}}{-1 - i\sqrt{3}}.\end{aligned}$$

You can divide complex numbers by using $\dfrac{1}{a+bi} = \dfrac{a-bi}{a^2+b^2}$. So

$$\begin{aligned}k_2 = \frac{-1 + i\sqrt{3}}{-1 - i\sqrt{3}} = (-1 + i\sqrt{3})\left(\frac{-1 + i\sqrt{3}}{4}\right) &= \frac{(-1 + i\sqrt{3})(-1 + i\sqrt{3})}{4}\\ = \frac{1 - i\sqrt{3} - i\sqrt{3} - 3}{4} = \frac{-2 - 2i\sqrt{3}}{4} &= \frac{-1 - i\sqrt{3}}{2}.\end{aligned}$$

Now we have $k_1 = 1$, $k_2 = \dfrac{-1 - i\sqrt{3}}{2}$, and since $k_3 = i\sqrt{3} + k_2$, we get

$$k_3 = \frac{-1 - i\sqrt{3}}{2} + i\sqrt{3} = \frac{-1 - i\sqrt{3}}{2} + \frac{2i\sqrt{3}}{2} = \frac{-1 - i\sqrt{3} + 2i\sqrt{3}}{2} = \frac{-1 + i\sqrt{3}}{2}.$$

Therefore the eigenvector associated to $\lambda = i\sqrt{3}$ is

$$K_+ = \begin{bmatrix} 1 \\ \frac{-1-i\sqrt{3}}{2} \\ \frac{-1+i\sqrt{3}}{2} \end{bmatrix},$$

which we can multiply by 2 to get

$$K_+ = \begin{bmatrix} 2 \\ -1 - i\sqrt{3} \\ -1 + i\sqrt{3} \end{bmatrix}$$

to avoid excessive fractions. Next we need to find the eigenvector associated to $\lambda = -i\sqrt{3}$, which we will call K_-. But remember: K_- and K_+ are *conjugates*! Meaning

$$K_- = \overline{K_+} = \overline{\begin{bmatrix} 2 \\ -1 - i\sqrt{3} \\ -1 + i\sqrt{3} \end{bmatrix}} = \begin{bmatrix} \overline{2} \\ \overline{-1 - i\sqrt{3}} \\ \overline{-1 + i\sqrt{3}} \end{bmatrix} = \begin{bmatrix} 2 \\ -1 + i\sqrt{3} \\ -1 - i\sqrt{3} \end{bmatrix}.$$

(Remember $\overline{a + bi} = a - bi$, so the real part a is left alone, but the imaginary part b has its sign flipped.) Now we can use

$$K_+ = \begin{bmatrix} 2 \\ -1 - i\sqrt{3} \\ -1 + i\sqrt{3} \end{bmatrix} \qquad \& \qquad K_- = \begin{bmatrix} 2 \\ -1 + i\sqrt{3} \\ -1 - i\sqrt{3} \end{bmatrix}$$

to make

$$B_1 = \frac{1}{2}(K_+ + K_-)$$

and

$$B_2 = \frac{i}{2}(-K_+ + K_-).$$

We get

$$B_1 = \frac{1}{2}\begin{bmatrix} 4 \\ -2 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$$

and

$$B_2 = \frac{i}{2}\left(\begin{bmatrix} -2 \\ 1 + i\sqrt{3} \\ 1 - i\sqrt{3} \end{bmatrix} + \begin{bmatrix} 2 \\ -1 + i\sqrt{3} \\ -1 - i\sqrt{3} \end{bmatrix}\right) = \frac{i}{2}\begin{bmatrix} 0 \\ 2i\sqrt{3} \\ -2i\sqrt{3} \end{bmatrix} = \begin{bmatrix} 0 \\ -\sqrt{3} \\ \sqrt{3} \end{bmatrix}.$$

Now we can build two new solutions using

$$Y_2 = B_1 e^{ax}\cos(bx) - B_2 e^{ax}\sin(bx)$$

and

$$Y_3 = B_2 e^{ax}\cos(bx) + B_1 e^{ax}\sin(bx)$$

where $a \pm bi$ are the eigenvalues. In our case, the eigenvalues are $\lambda = \pm i\sqrt{3}$, so $a = 0$ and $b = \sqrt{3}$. So we don't need the e^{ax} part in this case, because $e^{0x} = 1$.

So we get

$$Y_2 = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}\cos(\sqrt{3}x) - \begin{bmatrix} 0 \\ -\sqrt{3} \\ \sqrt{3} \end{bmatrix}\sin(\sqrt{3}x)$$

and

$$Y_3 = \begin{bmatrix} 0 \\ -\sqrt{3} \\ \sqrt{3} \end{bmatrix}\cos(\sqrt{3}x) + \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}\sin(\sqrt{3}x).$$

Putting this all together with $Y_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ from earlier, we can build the general solution

$$Y = c_1\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2\left(\begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}\cos(\sqrt{3}x) - \begin{bmatrix} 0 \\ -\sqrt{3} \\ \sqrt{3} \end{bmatrix}\sin(\sqrt{3}x)\right) + c_3\left(\begin{bmatrix} 0 \\ -\sqrt{3} \\ \sqrt{3} \end{bmatrix}\cos(\sqrt{3}x) + \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}\sin(\sqrt{3}x)\right)$$

which we can condense into

$$Y = \begin{bmatrix} c_1 + 2c_2\cos(\sqrt{3}x) + 2c_3\sin(\sqrt{3}x) \\ c_1 - c_2\cos(\sqrt{3}x) + \sqrt{3}c_2\sin(\sqrt{3}x) - \sqrt{3}c_3\cos(\sqrt{3}x) - c_3\sin(\sqrt{3}x) \\ c_1 - c_2\cos(\sqrt{3}x) - \sqrt{3}c_2\sin(\sqrt{3}x) + \sqrt{3}c_3\cos(\sqrt{3}x) - c_3\sin(\sqrt{3}x). \end{bmatrix}$$

So our general solution for rock, paper, scissors (r, p, and s) will be

$$\begin{aligned} r(x) &= c_1 + 2c_2\cos(\sqrt{3}x) + 2c_3\sin(\sqrt{3}x) \\ p(x) &= c_1 + (-c_2 - \sqrt{3}c_3)\cos(\sqrt{3}x) + (\sqrt{3}c_2 - c_3)\sin(\sqrt{3}x) \\ s(x) &= c_1 + (-c_2 + \sqrt{3}c_3)\cos(\sqrt{3}x) + (-\sqrt{3}c_2 - c_3)\sin(\sqrt{3}x), \end{aligned}$$

and will vary based on the parameters c_1, c_2, and c_3. You can see how the graphs of these functions look on Desmos here. (You can control the parameters c_1, c_2, and c_3 on the sidebar.)

2 Five Functions

Now let's expand this to the game *Rock, Paper, Scissors, Lizard, Spock.* The rules are:

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rock beats scissor and lizard :(
paper beats rock and Spock
scissors beats paper and lizard :(
lizard beats paper and Spock
Spock beats scissors and rock
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We can set up a similar set of differential equations as before. We will have to use N for Spock (because of Leonard Nimoy) because S is taken by the scissors and L is taken by the lizard.

$$\begin{aligned} R' &= -P + S + L - N \\ P' &= R - S - L + N \\ S' &= -R + P + L - N \\ L' &= -R + P - S + N \\ N' &= R - P + S - L \end{aligned}$$

which can be converted into a matrix equation

$$\begin{bmatrix} R' \\ P' \\ S' \\ L' \\ N' \end{bmatrix} = \begin{bmatrix} 0 & -1 & 1 & 1 & -1 \\ 1 & 0 & -1 & -1 & 1 \\ -1 & 1 & 0 & 1 & -1 \\ -1 & 1 & -1 & 0 & 1 \\ 1 & -1 & 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} R \\ P \\ S \\ L \\ N \end{bmatrix}.$$

Using a calculator, one can find the five eigenvalues of this matrix are

$$\begin{aligned} \lambda_0 &= 0 \\ \lambda_1 &= i\sqrt{5 + 2\sqrt{5}} \\ -\lambda_1 &= -i\sqrt{5 + 2\sqrt{5}} \\ \lambda_2 &= i\sqrt{5 - 2\sqrt{5}} \\ -\lambda_2 &= -i\sqrt{5 - 2\sqrt{5}} \end{aligned}$$

and the eigenvectors are written below. I made a graph of the solutions on Desmos so you can see what they look like.

The eigenvectors are

$$K_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \; K_1^+ = \begin{bmatrix} \sqrt{5} - 1 + i\left(\sqrt{\sqrt{20}+10}\right) \\ \sqrt{5} - 1 - i\left(\sqrt{\sqrt{20}+10}\right) \\ -\sqrt{5} - 1 + i\left(\sqrt{-\sqrt{20}+10}\right) \\ -\sqrt{5} - 1 - i\left(\sqrt{-\sqrt{20}+10}\right) \\ 4 \end{bmatrix}, \; K_1^- \begin{bmatrix} \sqrt{5} - 1 - i\left(\sqrt{\sqrt{20}+10}\right) \\ \sqrt{5} - 1 + i\left(\sqrt{\sqrt{20}+10}\right) \\ -\sqrt{5} - 1 - i\left(\sqrt{-\sqrt{20}+10}\right) \\ -\sqrt{5} - 1 + i\left(\sqrt{-\sqrt{20}+10}\right) \\ 4 \end{bmatrix},$$

$$K_2^+ = \begin{bmatrix} -\sqrt{5} - 1 - i\left(\sqrt{-\sqrt{20}+10}\right) \\ -\sqrt{5} - 1 + i\left(\sqrt{-\sqrt{20}+10}\right) \\ \sqrt{5} - 1 + i\left(\sqrt{\sqrt{20}+10}\right) \\ \sqrt{5} - 1 - i\left(\sqrt{\sqrt{20}+10}\right) \\ 4 \end{bmatrix}, \; K_2^- \begin{bmatrix} -\sqrt{5} - 1 + i\left(\sqrt{-\sqrt{20}+10}\right) \\ -\sqrt{5} - 1 - i\left(\sqrt{-\sqrt{20}+10}\right) \\ \sqrt{5} - 1 - i\left(\sqrt{\sqrt{20}+10}\right) \\ \sqrt{5} - 1 + i\left(\sqrt{\sqrt{20}+10}\right) \\ 4 \end{bmatrix}.$$

Thus we can use conjugate eigenvectors to make real vectors

$$B_1 = \frac{1}{2}(K_1^+ + K_1^-) = \begin{bmatrix} \sqrt{5} - 1 \\ \sqrt{5} - 1 \\ -\sqrt{5} - 1 \\ -\sqrt{5} - 1 \\ 4 \end{bmatrix},$$

$$C_1 = \frac{i}{2}(-K_1^+ + K_1^-) = \begin{bmatrix} \sqrt{\sqrt{20}+10} \\ -\sqrt{\sqrt{20}+10} \\ \sqrt{-\sqrt{20}+10} \\ -\sqrt{-\sqrt{20}+10} \\ 0 \end{bmatrix},$$

$$B_2 = \frac{1}{2}(K_2^+ + K_2^-) = \begin{bmatrix} -\sqrt{5} - 1 \\ -\sqrt{5} - 1 \\ \sqrt{5} - 1 \\ \sqrt{5} - 1 \\ 4 \end{bmatrix},$$

$$C_2 = \frac{i}{2}(-K_2^+ + K_2^-) = \begin{bmatrix} -\sqrt{-\sqrt{20}+10} \\ \sqrt{-\sqrt{20}+10} \\ \sqrt{\sqrt{20}+10} \\ -\sqrt{\sqrt{20}+10} \\ 0 \end{bmatrix}.$$

Thus we have the five generating solutions:

$$\begin{aligned}
Y_1 &= \vec{1} \\
Y_2 &= B_1 \cos\left(x\sqrt{5+2\sqrt{5}}\right) - C_1 \sin\left(x\sqrt{5+2\sqrt{5}}\right) \\
Y_3 &= C_1 \cos\left(x\sqrt{5+2\sqrt{5}}\right) + B_1 \sin\left(x\sqrt{5+2\sqrt{5}}\right) \\
Y_4 &= B_2 \cos\left(x\sqrt{5-2\sqrt{5}}\right) - C_2 \sin\left(x\sqrt{5-2\sqrt{5}}\right) \\
Y_5 &= C_2 \cos\left(x\sqrt{5-2\sqrt{5}}\right) + B_2 \sin\left(x\sqrt{5-2\sqrt{5}}\right)
\end{aligned}$$

and we may write a general solution of the form

$$\begin{bmatrix} R \\ P \\ S \\ L \\ N \end{bmatrix} = c_1Y_1 + c_2Y_2 + c_3Y_3 + c_4Y_4 + c_5Y_5.$$

3 Seven Functions

Let's look at the following system of differential equations.

$$\begin{bmatrix} y_1' \\ y_2' \\ y_3' \\ y_4' \\ y_5' \\ y_6' \\ y_7' \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & 0 & -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & 0 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 0 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & 0 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & 0 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 & 0 \end{bmatrix}}_{A} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \end{bmatrix}$$

The matrix A has characteristic polynomial

$$c_A(x) = -x(x^6 + 21x^4 + 35x^2 + 7).$$

According to LMFDB, the field $\mathrm{spl}(x^6 + 21x^4 + 35x^2 + 7) = \mathbb{Q}(\zeta_7)$, and so we can write the eigenvalues of A as rational linear combinations of the seventh roots of unity. They are as follows.

1. $\lambda_0 = 0$
2. $\lambda_1 = 2\zeta_7^3 + 2\zeta_7^2 + 2\zeta_7 + 1$

3. $\overline{\lambda_1} = -\lambda_1$
4. $\lambda_2 = 2\zeta_7^5 + 2\zeta_7^2 + 3\zeta_7 + 1$
5. $\overline{\lambda_2} = -\lambda_2$
6. $\lambda_3 = 2\zeta_7^5 + 2\zeta_7^4 + 2\zeta_7 + 1$
7. $\overline{\lambda_3} = -\lambda_3$

These eigenvalues can be given exact values as follows.

1. $\lambda_0 = 0$
2. $\lambda_1 = 2i(\sin(6\pi/7) + \sin(4\pi/7) + \sin(2\pi/7))$
3. $-\lambda_1 = -2i(\sin(6\pi/7) + \sin(4\pi/7) + \sin(2\pi/7))$
4. $\lambda_2 = 2i(\sin(10\pi/7) + \sin(6\pi/7) + \sin(2\pi/7))$
5. $-\lambda_2 = -2i(\sin(10\pi/7) + \sin(6\pi/7) + \sin(2\pi/7))$
6. $\lambda_3 = 2i(\sin(10\pi/7) + \sin(8\pi/7) + \sin(2\pi/7))$
7. $-\lambda_3 = -2i(\sin(10\pi/7) + \sin(8\pi/7) + \sin(2\pi/7))$

So we can write three nontrivial frequencies for this system of equations:

$$\begin{aligned}
F_1 &= 2(\sin(6\pi/7) + \sin(4\pi/7) + \sin(2\pi/7)), \\
F_2 &= 2(\sin(10\pi/7) + \sin(6\pi/7) + \sin(2\pi/7)), \\
F_3 &= 2(\sin(10\pi/7) + \sin(8\pi/7) + \sin(2\pi/7)).
\end{aligned}$$

We can also use $\mathbb{Q}(\zeta_7)$ to find exact expressions for the eigenvectors.

$$K_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix},$$

$$K_1 = \begin{bmatrix} \zeta_7^4 \\ \zeta_7 \\ \zeta_7^5 \\ \zeta_7^2 \\ \zeta_7^6 \\ \zeta_7^3 \\ 1 \end{bmatrix}, \overline{K}_1 = \begin{bmatrix} \zeta_7^3 \\ \zeta_7^6 \\ \zeta_7^2 \\ \zeta_7^5 \\ \zeta_7 \\ \zeta_7^4 \\ 1 \end{bmatrix},$$

$$K_2 = \begin{bmatrix} \zeta_7^6 \\ \zeta_7^5 \\ \zeta_7^4 \\ \zeta_7^3 \\ \zeta_7^2 \\ \zeta_7 \\ 1 \end{bmatrix}, \overline{K}_2 = \begin{bmatrix} \zeta_7 \\ \zeta_7^2 \\ \zeta_7^3 \\ \zeta_7^4 \\ \zeta_7^5 \\ \zeta_7^6 \\ 1 \end{bmatrix},$$

$$K_3 = \begin{bmatrix} \zeta_7^2 \\ \zeta_7^4 \\ \zeta_7^6 \\ \zeta_7 \\ \zeta_7^3 \\ \zeta_7^5 \\ 1 \end{bmatrix}, \overline{K}_3 = \begin{bmatrix} \zeta_7^5 \\ \zeta_7^3 \\ \zeta_7 \\ \zeta_7^6 \\ \zeta_7^4 \\ \zeta_7^2 \\ 1 \end{bmatrix}.$$

This allows to construct the real vectors

$$R_1 = \frac{1}{2}(K_1 + \overline{K}_1) = \begin{bmatrix} \cos(6\pi/7) \\ \cos(2\pi/7) \\ \cos(4\pi/7) \\ \cos(4\pi/7) \\ \cos(2\pi/7) \\ \cos(6\pi/7) \\ 1 \end{bmatrix},$$

$$J_1 = \frac{i}{2}(-K_1 + \overline{K}_1) = \begin{bmatrix} \sin(6\pi/7) \\ \sin(12\pi/7) \\ \sin(4\pi/7) \\ \sin(10\pi/7) \\ \sin(2\pi/7) \\ \sin(8\pi/7) \\ 0 \end{bmatrix},$$

$$R_2 = \frac{1}{2}(K_2 + \overline{K}_2) = \begin{bmatrix} \cos(2\pi/7) \\ \cos(4\pi/7) \\ \cos(6\pi/7) \\ \cos(6\pi/7) \\ \cos(4\pi/7) \\ \cos(2\pi/7) \\ 1 \end{bmatrix},$$

$$J_2 = \frac{i}{2}(-K_2 + \overline{K}_2) = \begin{bmatrix} \sin(2\pi/7) \\ \sin(4\pi/7) \\ \sin(6\pi/7) \\ \sin(8\pi/7) \\ \sin(10\pi/7) \\ \sin(12\pi/7) \\ 0 \end{bmatrix},$$

$$R_3 = \frac{1}{2}(K_3 + \overline{K}_3) = \begin{bmatrix} \cos(4\pi/7) \\ \cos(6\pi/7) \\ \cos(2\pi/7) \\ \cos(2\pi/7) \\ \cos(6\pi/7) \\ \cos(4\pi/7) \\ 1 \end{bmatrix},$$

$$J_3 = \frac{i}{2}(-K_3 + \overline{K}_3) = \begin{bmatrix} \sin(10\pi/7) \\ \sin(6\pi/7) \\ \sin(2\pi/7) \\ \sin(12\pi/7) \\ \sin(8\pi/7) \\ \sin(4\pi/7) \\ 0 \end{bmatrix}.$$

Then we can construct the seven generating solutions.

$$\begin{aligned} Y_0 &= \vec{1} \\ Y_1 &= R_1 \cos(F_1 x) - J_1 \sin(F_1 x) \\ Z_1 &= J_1 \cos(F_1 x) + R_1 \sin(F_1 x) \\ Y_2 &= R_2 \cos(F_2 x) - J_2 \sin(F_2 x) \\ Z_2 &= J_2 \cos(F_2 x) + R_2 \sin(F_2 x) \\ Y_3 &= R_3 \cos(F_3 x) - J_3 \sin(F_3 x) \\ Z_3 &= J_3 \cos(F_3 x) + R_3 \sin(F_3 x) \end{aligned}$$

We may write a general solution of the form

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \end{bmatrix} = c_0 Y_0 + c_1 Y_1 + d_1 Z_1 + c_2 Y_2 + d_2 Z_2 + c_3 Y_3 + d_3 Z_3.$$

Here is a graph I made on Desmos so you can see how the solutions look.