

Let  $P_1, \dots, P_7 \in \mathbb{P}_k^2$ . First note that  $\binom{8+2}{2} = 45$ , and so 44 conditions will determine a unique octic. Now consider the six triple points  $P_1, \dots, P_6$ . Six triple points impose  $6 * \binom{3+1}{2} = 6 * 6 = 36$  conditions, and choosing  $P_7$  as a double point imposes an extra  $\binom{2+1}{2} = 3$  conditions for a total of 39. Thus there is a  $45 - 39 = 6$ -dimensional vector space of octics that have triple points at  $P_1, \dots, P_6$  and a double point at  $P_7$ . The six generating octics are simple to construct:

$$\begin{aligned} O_1 &= C(1, 2, 3, 4, 5) C(2, 3, 4, 5, 6) C(1, 6, 3, 4, 7) C(1, 6, 2, 5, 7) \\ O_2 &= C(1, 2, 3, 4, 6) C(1, 3, 4, 5, 6) C(2, 5, 7, 1, 3) C(2, 5, 7, 4, 6) \\ O_3 &= C(1, 2, 3, 5, 6) C(1, 2, 4, 5, 6) C(3, 4, 1, 2, 7) C(3, 4, 7, 5, 6) \\ O_4 &= C(1, 2, 3, 4, 6) C(2, 3, 4, 5, 6) C(1, 5, 7, 2, 3) C(1, 6, 7, 4, 5) \\ O_5 &= C(1, 3, 4, 5, 6) C(1, 2, 3, 4, 5) C(2, 6, 7, 1, 5) C(2, 6, 7, 3, 4) \\ O_6 &= C(1, 2, 3, 5, 6) C(2, 3, 4, 5, 6) C(1, 4, 7, 2, 3) C(1, 4, 7, 5, 6) \end{aligned}$$

where each octic is a product of four conics determined by the five specified points.

In order to upgrade  $P_7$  from a double point to a triple point, we need to choose coefficients  $c_1, \dots, c_6$  that produces an adequate linear combination of  $O_1, \dots, O_6$ . Since seven triple points induces 42 conditions, we need only use the first four octics  $O_1, \dots, O_4$ .

We want a linear combination

$$O = c_1 O_1 + c_2 O_2 + c_3 O_3 + c_4 O_4$$

where  $O_{xy}(P_7) = O_{xz}(P_7) = O_{yz}(P_7) = 0$ . This is sufficient since  $P_7$  is already a double point. That is, we are already given that  $O_x(P_7) = O_y(P_7) = O_z(P_7) = 0$ , and so if the desired conditions are fulfilled, then  $O_x(P_7) = xO_{xx}(P_7) + yO_{xy}(P_7) + zO_{xz}(P_7) = 0$  and we must have  $O_{xx}(P_7) = 0$  as well. Same goes for  $O_{yy}(P_7)$  and  $O_{zz}(P_7)$ .

We want

$$\begin{aligned} c_1 O_{1xy}(P_7) + c_2 O_{2xy}(P_7) + c_3 O_{3xy}(P_7) + c_4 O_{4xy}(P_7) &= 0 \\ c_1 O_{1xz}(P_7) + c_2 O_{2xz}(P_7) + c_3 O_{3xz}(P_7) + c_4 O_{4xz}(P_7) &= 0 \\ c_1 O_{1yz}(P_7) + c_2 O_{2yz}(P_7) + c_3 O_{3yz}(P_7) + c_4 O_{4yz}(P_7) &= 0 \end{aligned}$$

which can be turned into the following matrix equation:

$$\begin{pmatrix} O_{1xy} & O_{2xy} & O_{3xy} & O_{4xy} \\ O_{1xz} & O_{2xz} & O_{3xz} & O_{4xz} \\ O_{1yz} & O_{2yz} & O_{3yz} & O_{4yz} \end{pmatrix} (P_7) \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

So

$$\vec{c} \in \ker \begin{pmatrix} O_{1xy} & O_{2xy} & O_{3xy} & O_{4xy} \\ O_{1xz} & O_{2xz} & O_{3xz} & O_{4xz} \\ O_{1yz} & O_{2yz} & O_{3yz} & O_{4yz} \end{pmatrix} (P_7)$$

and so we can take

$$\vec{c} = \begin{pmatrix} \det \begin{pmatrix} O_{2xy} & O_{3xy} & O_{4xy} \\ O_{2xz} & O_{3xz} & O_{4xz} \\ O_{2yz} & O_{3yz} & O_{4yz} \end{pmatrix} \\ -\det \begin{pmatrix} O_{1xy} & O_{3xy} & O_{4xy} \\ O_{1xz} & O_{3xz} & O_{4xz} \\ O_{1yz} & O_{3yz} & O_{4yz} \end{pmatrix} \\ \det \begin{pmatrix} O_{1xy} & O_{2xy} & O_{4xy} \\ O_{1xz} & O_{2xz} & O_{4xz} \\ O_{1yz} & O_{2yz} & O_{4yz} \end{pmatrix} \\ -\det \begin{pmatrix} O_{1xy} & O_{2xy} & O_{3xy} \\ O_{1xz} & O_{2xz} & O_{3xz} \\ O_{1yz} & O_{2yz} & O_{3yz} \end{pmatrix} \end{pmatrix} (P_7).$$

Then  $\vec{c} \in \ker \begin{pmatrix} O_{1xy} & O_{2xy} & O_{3xy} & O_{4xy} \\ O_{1xz} & O_{2xz} & O_{3xz} & O_{4xz} \\ O_{1yz} & O_{2yz} & O_{3yz} & O_{4yz} \end{pmatrix} (P_7)$  because for example

$$\begin{aligned} & O_{1xy} \det \begin{pmatrix} O_{2xy} & O_{3xy} & O_{4xy} \\ O_{2xz} & O_{3xz} & O_{4xz} \\ O_{2yz} & O_{3yz} & O_{4yz} \end{pmatrix} - O_{2xy} \det \begin{pmatrix} O_{1xy} & O_{3xy} & O_{4xy} \\ O_{1xz} & O_{3xz} & O_{4xz} \\ O_{1yz} & O_{3yz} & O_{4yz} \end{pmatrix} \\ & + O_{3xy} \det \begin{pmatrix} O_{1xy} & O_{2xy} & O_{4xy} \\ O_{1xz} & O_{2xz} & O_{4xz} \\ O_{1yz} & O_{2yz} & O_{4yz} \end{pmatrix} - O_{4xy} \det \begin{pmatrix} O_{1xy} & O_{2xy} & O_{3xy} \\ O_{1xz} & O_{2xz} & O_{3xz} \\ O_{1yz} & O_{2yz} & O_{3yz} \end{pmatrix} \\ & = \det \begin{pmatrix} O_{1xy} & O_{2xy} & O_{3xy} & O_{4xy} \\ O_{1xz} & O_{2xz} & O_{3xz} & O_{4xz} \\ O_{1yz} & O_{2yz} & O_{3yz} & O_{4yz} \end{pmatrix} = 0. \end{aligned}$$

The same applies to the other two rows.

So finally

$$(O_1 \ O_2 \ O_3 \ O_4) \vec{c}$$

is an octic polynomial that vanishes with multiplicity 3 at  $P_1, \dots, P_7$ .

You can find a Desmos gadget illustrating this octic here, but be warned that it is very slow to load.