This will provide an explanation behind these Desmos demonstrations:
`https://www.desmos.com/calculator/j7sw3ezr2h` and
`https://www.desmos.com/calculator/nkdgbhhyke`.

First we have five points $P_1, \ldots, P_5$. Let $P_i = (a_i, b_i, c_i)$ for $1 \leq i \leq 5$. First we will find an automorphism of the plane $A : \mathbb{P}^2 \to \mathbb{P}^2$ such that

$$\begin{aligned} A(P_1) &= (1,0,0), \\ A(P_2) &= (0,1,0), \\ A(P_3) &= (0,0,1), \\ \text{and } A(P_4) &= (1,1,1). \end{aligned}$$

My first step of this was to find a matrix that would just do the first three things. I got

$$M = \begin{pmatrix} (b_2c_3 - b_3c_2) & (a_3c_2 - a_2c_3) & (a_2b_3 - a_3b_2) \\ (b_1c_3 - b_3c_1) & (a_3c_1 - a_1c_3) & (a_1b_3 - a_3b_1) \\ (b_1c_2 - b_2c_1) & (a_2c_1 - a_1c_2) & (a_1b_2 - a_2b_1) \end{pmatrix}.$$

To get a function that also sends P_4 to $(1,1,1)$, let us first define the functions

$$\begin{aligned} L_{23}(x,y,z) &= (b_2c_3 - b_3c_2)x + (a_3c_2 - a_2c_3)y + (a_2b_3 - a_3b_2)z, \\ L_{13}(x,y,z) &= (b_1c_3 - b_3c_1)x + (a_3c_1 - a_1c_3)y + (a_1b_3 - a_3b_1)z, \\ L_{12}(x,y,z) &= (b_1c_2 - b_2c_1)x + (a_2c_1 - a_1c_2)y + (a_1b_2 - a_2b_1)z. \end{aligned}$$

In other words,

$$M(x,y,z) = (L_{23}(x,y,z), L_{13}(x,y,z), L_{12}(x,y,z)).$$

Then define $f = 1/L_{23}(P_4)$, $g = 1/L_{13}(P_4)$, and $h = 1/L_{12}(P_4)$. Then

$$A = \begin{pmatrix} (b_2c_3 - b_3c_2)f & (a_3c_2 - a_2c_3)f & (a_2b_3 - a_3b_2)f \\ (b_1c_3 - b_3c_1)g & (a_3c_1 - a_1c_3)g & (a_1b_3 - a_3b_1)g \\ (b_1c_2 - b_2c_1)h & (a_2c_1 - a_1c_2)h & (a_1b_2 - a_2b_1)h \end{pmatrix}.$$

Then $A(P_5) = (L_{23}(P_5)f, L_{13}(P_5)g, L_{12}(P_5)h)$. Now we will apply the quadratic transformation with base points at $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$ (also called the *Cremona* transformation) $\varphi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ given by $\varphi(a,b,c) = (bc, ac, ab)$. Then φ preserves $(1,1,1)$ and φ sends $A(P_5)$ to

$$\varphi(A(P_5)) = (L_{13}(P_5)L_{12}(P_5)gh, L_{23}(P_5)L_{12}(P_5)fh, L_{23}(P_5)L_{13}(P_5)fg).$$

Define

$$\begin{aligned} Q_1 &= L_{23}(P_5)L_{12}(P_5)fh - L_{23}(P_5)L_{13}(P_5)fg \\ Q_2 &= L_{23}(P_5)L_{13}(P_5)fg - L_{13}(P_5)L_{12}(P_5)gh \\ Q_3 &= L_{13}(P_5)L_{12}(P_5)gh - L_{23}(P_5)L_{12}(P_5)fh. \end{aligned}$$

Then the line connecting $(1,1,1)$ and $\varphi(A(P_5))$ is given by the polynomial $Q_1x+Q_2y+Q_3z$. After applying φ again, we get a conic $Q_1yz+Q_2xz+Q_3xy$ that goes through $\varphi(\varphi(A(P_5))) = P_5$, $(1,1,1)$, $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$. Now all we need to do is apply A^{-1} to this conic and we get

$$Q_1L_{13}(x,y,z)gL_{12}(x,y,z)h + Q_2L_{23}(x,y,z)fL_{12}(x,y,z)h + Q_3L_{23}(x,y,z)fL_{13}(x,y,z)g.$$

This is the conic that goes through $P_1,\ldots,P_5$.

Now suppose we have four points $P_1, \ldots, P_4$ and a line T. We will show how to construct the two conics that go through the four points and is tangent to T.

First let us first determine how to find the points on a conic C whose line connecting to $(1,1,1)$ is tangent to C. Let C be given by the expression $qx^2+rxy+sxz+ty^2+uyz+vz^2$ and let $P = (a,b,c) \in C$. Then the line tangent to C at P is given by

$$(2qa + rb + sc)x + (ra + 2tb + uc)y + (sa + ub + 2vc)z.$$

For $(1,1,1)$ to be on this line, (a,b,c) must satisfy

$$(2q + r + s)a + (r + 2t + u)b + (s + u + 2v)c = \alpha a + \beta b + \gamma c = 0.$$

In other words, P must be where the line $L = \alpha x + \beta y + \gamma z$ intersects C. There are at most 2 such points. Let

$$\begin{aligned} \delta &= q\gamma^2 - s\alpha\gamma + v\alpha^2, \\ \varepsilon &= r\gamma^2 - s\beta\gamma - u\alpha\gamma + 2v\alpha\beta, \\ \psi &= t\gamma^2 - u\beta\gamma + v\beta^2. \end{aligned}$$

The the intersection with the line L and the conic C occur at the solutions to $\delta x^2 + \varepsilon xy + \psi y^2$. These solutions are $(-\varepsilon \pm \sqrt{\varepsilon^2 - 4\delta\psi}, 2\delta)$. To be on the line $L = \alpha x + \beta y + \gamma z$, we get $z = \dfrac{-\alpha(-\varepsilon \pm \sqrt{\varepsilon^2 - 4\delta\psi}) - \beta(2\delta)}{\gamma}$. Thus our points are

$$\left(-\gamma\varepsilon \pm \gamma\sqrt{\varepsilon^2 - 4\delta\psi}, 2\gamma\delta, -\alpha\left(-\varepsilon \pm \sqrt{\varepsilon^2 - 4\delta\psi}\right) - 2\beta\delta\right).$$

Now let $T = ax + by + cz$. Note

$$A^{-1} = \begin{pmatrix} -a_1/f & a_2/g & -a_3/h \\ -b_1/f & b_2/g & -b_3/h \\ -c_1/f & c_2/g & -c_3/h \end{pmatrix}.$$

Then

$$A(T) = a\left(-\frac{a_1}{f}x + \frac{a_2}{g}y - \frac{a_3}{h}z\right) + b\left(-\frac{b_1}{f}x + \frac{b_2}{g}y - \frac{b_3}{h}z\right) + c\left(-\frac{c_1}{f}x + \frac{c_2}{g}y - \frac{c_3}{h}z\right).$$

Then

$$\varphi(A(T)) = a\left(-\frac{a_1}{f}yz + \frac{a_2}{g}xz - \frac{a_3}{h}xy\right) + b\left(-\frac{b_1}{f}yz + \frac{b_2}{g}xz - \frac{b_3}{h}xy\right) + c\left(-\frac{c_1}{f}yz + \frac{c_2}{g}xz - \frac{c_3}{h}xy\right)$$

and so

$$\begin{aligned}
q &= 0\\
r &= -\frac{aa_3}{h} - \frac{bb_3}{h} - \frac{cc_3}{h}\\
s &= \frac{aa_2}{g} + \frac{bb_2}{g} + \frac{cc_2}{g}\\
t &= 0\\
u &= -\frac{aa_1}{f} - \frac{bb_1}{f} - \frac{cc_1}{f}\\
v &= 0.
\end{aligned}$$

Therefore

$$\begin{aligned}
\alpha &= r + s\\
\beta &= r + u\\
\gamma &= s + u\\
\delta &= -s\alpha\gamma\\
\varepsilon &= r\gamma^2 - s\beta\gamma - u\alpha\gamma\\
\psi &= -u\beta\gamma.
\end{aligned}$$

With these equalities in mind, define

$$\begin{aligned}
\eta_1 &= -\gamma\varepsilon + \gamma\sqrt{\varepsilon^2 - 4\delta\psi}\\
\eta_2 &= -\gamma\varepsilon - \gamma\sqrt{\varepsilon^2 - 4\delta\psi}\\
\zeta &= 2\gamma\delta\\
\mu_1 &= -\alpha\left(-\varepsilon + \sqrt{\varepsilon^2 - 4\delta\psi}\right) - 2\beta\delta\\
\mu_2 &= -\alpha\left(-\varepsilon - \sqrt{\varepsilon^2 - 4\delta\psi}\right) - 2\beta\delta
\end{aligned}$$

and so

$$\begin{aligned}
\Phi_1 &= \left(-\gamma\varepsilon + \gamma\sqrt{\varepsilon^2 - 4\delta\psi}, 2\gamma\delta, -\alpha\left(-\varepsilon + \sqrt{\varepsilon^2 - 4\delta\psi}\right) - 2\beta\delta\right) = (\eta_1, \zeta, \mu_1)\\
\Phi_2 &= \left(-\gamma\varepsilon - \gamma\sqrt{\varepsilon^2 - 4\delta\psi}, 2\gamma\delta, -\alpha\left(-\varepsilon - \sqrt{\varepsilon^2 - 4\delta\psi}\right) - 2\beta\delta\right) = (\eta_2, \zeta, \mu_2).
\end{aligned}$$

Now we must connect Φ_1 and Φ_2 to $(1,1,1)$ with the lines

$$\begin{aligned}
L_1 &= (\zeta - \mu_1)x + (\mu_1 - \eta_1)y + (\eta_1 - \zeta)z\\
L_2 &= (\zeta - \mu_2)x + (\mu_2 - \eta_2)y + (\eta_2 - \zeta)z.
\end{aligned}$$

Now we must apply the quadratic transformation φ to these lines to get

$$\begin{aligned}
\varphi(L_1) &= (\zeta - \mu_1)yz + (\mu_1 - \eta_1)xz + (\eta_1 - \zeta)xy\\
\varphi(L_2) &= (\zeta - \mu_2)yz + (\mu_2 - \eta_2)xz + (\eta_2 - \zeta)xy.
\end{aligned}$$

These are conics that contain $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, and $(1,1,1)$ and are tangent to $A(T)$. Let us write

$$\begin{aligned} R_1 &= \zeta - \mu_1 \\ R_2 &= \mu_1 - \eta_1 \\ R_3 &= \eta_1 - \zeta \\ S_1 &= \zeta - \mu_2 \\ S_2 &= \mu_2 - \eta_2 \\ S_3 &= \eta_2 - \zeta. \end{aligned}$$

Now we must apply A^{-1} to $\varphi(L_1)$ and $\varphi(L_2)$ to get

$$C_1 = R_1 L_{13}(x,y,z) g L_{12}(x,y,z) h + R_2 L_{23}(x,y,z) f L_{12}(x,y,z) h + R_3 L_{23}(x,y,z) f L_{13}(x,y,z) g$$
$$C_2 = S_1 L_{13}(x,y,z) g L_{12}(x,y,z) h + S_2 L_{23}(x,y,z) f L_{12}(x,y,z) h + S_3 L_{23}(x,y,z) f L_{13}(x,y,z) g.$$

These are the two conics that go through points $P_1, \ldots, P_4$ and are tangent to the line T.