Let X be an elliptic surface. Then X is isomorphic to $\mathbb{P}^2$ blown up at nine (not necessarily distinct) points $P_1, \dots, P_9$ in Cayley-Bacharach position (that is, $h^0(3\ell - P_1 - \cdots - P_9) = 2$). Let e_i denote the exceptional divisor for P_i and let K be the canonical divisor

$$-3\ell + e_1 + \cdots + e_9.$$

Denote by $K^\perp$ the group of divisors D on X satisfying $D.K = 0$. Note that

$$K^\perp = \langle \ell - e_1 - e_2 - e_3, e_1 - e_2, e_2 - e_3, \dots, e_8 - e_9 \rangle.$$

Denote by $\langle -2 \rangle$ the group generated by the effective (-2)-curves on X. Note that we have $\langle -2 \rangle \leq K^\perp$.

We know that the (-1)-curves on X are sections corresponding to linear combinations of the nine base points $P_1, \dots, P_9$, and also that each (-1)-curve corresponds to an element of the group $K^\perp / \langle -2 \rangle$.

From this information we can deduce the order of points on an elliptic curve E and identify points of infinite order.

Example 1. Consider the pencil of cubics spanned by $x^3 - xz^2$ and $y^3 - yz^2$. This pencil has four reducible fibers, two having three components and two having two components. This gives us the following ten effective (-2)-curves.

- $\ell - e_1 - e_2 - e_3$, $\ell - e_4 - e_5 - e_6$, $\ell - e_7 - e_8 - e_9$
- $\ell - e_1 - e_5 - e_9$, $2\ell - e_2 - e_3 - e_4 - e_6 - e_7 - e_8$
- $\ell - e_1 - e_4 - e_7$, $\ell - e_2 - e_5 - e_8$, $\ell - e_3 - e_6 - e_9$
- $\ell - e_3 - e_5 - e_7$, $2\ell - e_1 - e_2 - e_4 - e_6 - e_8 - e_9$

We can use Sage to show that $K^\perp / \langle -2 \rangle \cong \mathbb{Z} \times \mathbb{Z}$:

```
sage:  G = span(ZZ,[[1,-1,-1,-1,0,0,0,0,0,0],[0,1,-1,0,0,0,0,0,0,0],
[0,0,1,-1,0,0,0,0,0,0],[0,0,0,1,-1,0,0,0,0,0],[0,0,0,0,1,-1,0,0,0,0],
[0,0,0,0,0,1,-1,0,0,0],[0,0,0,0,0,0,1,-1,0,0],[0,0,0,0,0,0,0,1,-1,0],
[0,0,0,0,0,0,0,0,1,-1]])
sage:  H = G.span([[1,-1,-1,-1,0,0,0,0,0,0],[1,0,0,0,-1,-1,-1,0,0,0],
[1,0,0,0,0,0,0,-1,-1,-1],[1,-1,0,0,-1,0,0,-1,0,0],[1,0,-1,0,0,-1,0,0,-1,0],
[1,0,0,-1,0,0,-1,0,0,-1],[1,-1,0,0,0,-1,0,0,0,-1],[1,0,0,-1,0,-1,0,-1,0,0],
[2,0,-1,-1,-1,0,-1,-1,-1,0],[2,-1,-1,0,-1,0,-1,0,-1,-1]])
sage:  G / H
Finitely generated module V/W over Integer Ring with invariants (0, 0)
```

Therefore there are infinitely many sections in the elliptic surface given by this pencil, and so the subgroup generated by $P_1, \dots, P_9$ on any of the smooth fibers of the pencil under the elliptic group law has infinite order. (Specifically, $\langle P_1, \dots, P_9 \rangle \cong \mathbb{Z} \times \mathbb{Z}$.)