1 Introduction

Denote by $\mathrm{PGr}(k,n;q)$ the Grassmannian of projective k-planes in $\mathbb{P}^n_{\mathbb{F}_q}$ (that is, $\mathrm{PGr}(k,n;q) = \mathrm{Gr}(k+1,n+1;q)$, the Grassmannian of $k+1$-dimensional vector subspaces of $\mathbb{F}_q^n$).

Then

$$\#\mathrm{PGr}(k,n;q) = \binom{n+1}{k+1}_q := \frac{(1-q^{n+1})\cdots(1-q^{(n+1)-(k+1)+1})}{(1-q^{k+1})\cdots(1-q)}.$$

Let $\mathcal{A} = \mathrm{PGr}(k,n;q)$ and $\mathcal{B} = \mathrm{PGr}(h,n;q)$, with $k<h$. We take R to be the incidence relation $R = \{(\ell,m) \in \mathcal{A}\times\mathcal{B} : \ell \subseteq m\}$. Since each k-subspace is contained in $\binom{n-k}{h-k}_q$ subspaces of dimension h and each h-space contains $\binom{h+1}{k+1}_q$ subspaces of dimension k, we obtain a

$$\left(\left[\binom{n+1}{k+1}_q\right]_{\binom{n-k}{h-k}_q}, \left[\binom{n+1}{h+1}_q\right]_{\binom{h+1}{k+1}_q}\right)\text{-configuration.}$$

This is a *projective geometry configuration* and will be denoted $\mathrm{PG}(n,k,h;q)$. Note that this is a complete configuration of Grassmannians; $\mathcal{A}$ is all of $\mathrm{PGr}(k,n;q)$ and $\mathcal{B}$ is all of $\mathrm{PGr}(h,n;q)$.

We will define a *projective design* as a $\mathrm{PG}(n,0,h;q)$.

In general, a **design** is a (v_k,b_r)-configuration where, for any distinct $x,y\in\mathcal{A}$,

$$\#(R(x)\cap R(y)) = \lambda \neq 0 \text{ is constant.}$$

We will call λ the **type** of the design. A symmetric v_k-design of type λ has the additional property that, for any $x,y\in\mathcal{B}$,

$$\#(R(x)\cap R(y)) = \lambda.$$

The type of $\mathrm{PG}(n,0,h;q)$ is $\binom{n-1}{h-1}_q$.

2 Examples

A **linear** (v_k,b_r)-configuration over an infinite field K is a configuraiton in which the set $\mathcal{A}$ (resp. $\mathcal{B}$) is realized by a set of linear subspaces of dimension d_1 (resp. d_2) of the projective $\mathbb{P}^n_K$, and the relation R is an incidence relation $x\in R(y)$ if x intersects y along a subspace of fixed dimension s. The data $(\mathbb{P}^n_K, s; d_1, d_2)$ is the type of a linear realization.

For example, any finite linear configuration $\mathrm{PG}(n,k,h;q)$ is a linear configuration over $\overline{\mathbb{F}}_q$ of $(\mathbb{P}^n_{\overline{F}_q}, k; k, h)$.

One can perform the following operations upon a linear

2.1 Elliptic Embedding

Proposition 1. Let (E, x_0) be an elliptic curve with determined zero point x_0. Then for $n \in \mathbb{Z}$, the linear system $|nx_0|$ induces an embedding $\Phi_n : E \hookrightarrow \mathbb{P}^{n-1}$ in such a way that the image of every point in $(E, x_0)[n]$ is hyperosculating on $\Phi_n(E)$.

First let us observe the proposition for the specific curve $(E, x_0) = (V(x^3+y^3+z^3), (-1, 1, 0))$ for $n = 3$.

Then $\mathcal{L}(3x_0) = \langle 1, \tau_1/\tau_0, \tau_2/\tau_0\rangle$, where $\tau_0 = x + y$ is the flex tangent line through x_0, $\tau_1 = x + z$ is the flex tangent line through $(-1, 0, 1)$, and $\tau_2 = tx + z$ is the flex tangent line through $(-1, 0, t)$. So $|3x_0|$ defines the embedding $\Phi_3 : E \to \mathbb{P}^2$ given by

$$\Phi_3(x, y, z) = (1, (x + z)/(x + y), (tx + z)/(x + y)) = (x + y, x + z, tx + z).$$

This is just a linear change of coordinates in PGL(2), so flex points must map to flex points, which are hyperosculating.

In particular, x_0 maps to $(0, 1, t)$; and x_0 must be the only point of E that maps into the line $w_0 = 0$ (using w_0, w_1, w_2-variables for the codomain $\mathbb{P}^2$) because x_0 is the only point of E on $x + y = 0$, so $w_0 = 0$ is hyperosculating. By the same reasoning, $w_1 = 0$ is the hyperosculating line of $\Phi_3(-1, 0, 1)$, and $w_2 = 0$ is the hyperosculating line of $\Phi_3(-1, 0, t)$.

Now let us go up to the $n = 4$ case. Let

$$\mathcal{L}(4x_0) = \langle 1, r_1/\tau_0, \tau_1/\tau_0, r_1^2/\tau_0^2\rangle,$$

where r_1 is the line connecting the flex point $(-1, 0, 1)$ to $x_0 = (-1, 1, 0)$, so $r_1 = x + y + z$. Then $|4x_0|$ defines an embedding $\Phi_4 : E \to \mathbb{P}^3$ given by

$$\Phi_4(x, y, z) = ((x + y)^2, (x + y + z)(x + y), (x + y)(x + z), (x + y + z)^2).$$

We can see a Desmos graph of this embedding here.

Then note that x_0 seems to map to $(0, 0, 0, 0)$ under this map. In reality, $x_0 \mapsto (0^6, 0 * 0^3, 0 * 0^3, 0^2)$, so removing two orders of 0 gives us $\Phi_4(x_0) = (0, 0, 0, 1)$.

We can also see this by taking the limit

$$\lim_{\varepsilon\to 0} \Phi_4(-1, 1, \varepsilon) = \lim_{\varepsilon\to 0}(0, 0, 0, \varepsilon^2) = \lim_{\varepsilon\to 0}(0, 0, 0, 1) = (0, 0, 0, 1).$$

The hyperosculating plane to this point of $\Phi_4(E)$ is $w_0 = 0$, since x_0 is the only point of E on $\tau_0^2 = 0$.

Now let q be any 4-torsion point of (E, x_0): that is, $q \in (E, x_0)[4]$. Then there is a rational function Q such that

$$\operatorname{div}(Q) = 4q - 4x_0,$$

which means $Q \in \mathcal{L}(4x_0)$. Thus there are coefficients $A, B, C, D \in K$ where

$$Q = A + B(r_1/\tau_0) + C(\tau_1/\tau_0) + D(r_1/\tau_0)^2,$$

or

$$Q\tau_0^2 = A\tau_0^2 + Br_1\tau_0 + C\tau_1\tau_0 + Dr_1^2.$$

Then $Q\tau_0^2$ is a degree-2 function where

$$\operatorname{div}(Q\tau_0) = 4q + 2x_0,$$

so $\operatorname{ord}_q(Q\tau_0) = 4$. Thus the plane

$$V(Aw_0 + Bw_1 + Cw_2 + Dw_3) \subseteq \mathbb{P}^3_K$$

contains $\Phi_4(q)$, and meets $\Phi_4(E)$ **only** at $\Phi_4(E)$ (remember we had to account for two orders of 0's for x_0).

In general, if x_0 is not flex, we can take $\mathcal{L}(3x_0) = \langle 1, f_1/f_2, g_1/g_2\rangle$ where $\operatorname{ord}_{x_0}(f_1/f_2) = -2$ and $\operatorname{ord}_{x_0}(g_1/g_2) = -3$. Then whatever...

2.2 Modular Configurations

Let $p > 2$ be prime, and consider the group $G = (\mathbb{Z}/p\mathbb{Z})^2$. We take $\mathcal{A}$ to be the set of cosets for each subgroup of order p of G. We take $\mathcal{B}$ to be G itself.

Note that $\mathcal{A}$ is equivalent to lines through the affine space $\mathbb{A}^2_{\mathbb{F}_p}$: this is because each subgroup of order p is a line through the origin of $\mathbb{A}^2_p$, and then each coset shifts the line of the same slope.

Thus

$$\#\mathcal{A} = \frac{p^2(p^2-1)}{p(p-1)} = p(p+1).$$

Each line contains p points, and each line is on $p+1$ points, so we get a

$$(p(p+1)_p, p^2_{p+1})\text{-configuration.}$$

We can define an action of G on $\mathbb{P}^{p-1}_K$ the following way (assuming K has p distinct p^{th} roots of unity). We can define

$$(1,0)\cdot(x_0,\dots,x_{p-1}) = (x_{p-1}, x_0, \dots, x_{p-2}),$$

and

$$(0,1)\cdot(x_0,\dots,x_{p-1}) = (x_0, \zeta x_1, \zeta^2 x_2, \dots, \zeta^{p-1}x_{p-1}).$$

Then $G \hookrightarrow \operatorname{PGL}(p,K)$ and one can check using spectral theory that there are p fixed hyperplanes for each vector of $\{(1,0)\} \cup \{(n,1) : n \in \mathbb{Z}/p\mathbb{Z}\}$, and thus for each subgroup of order p of G.

Specifically, $(1,0)$ fixes all hyperplanes of the form

$$\begin{aligned}
x_0 + x_1 + x_2 + \cdots + x_{p-1} &= 0,\\
x_0 + \zeta x_1 + \zeta^2 x_2 + \cdots + \zeta^{p-1}x_{p-1} &= 0,\\
x_0 + \zeta^2 x_1 + \zeta^4 x_2 + \cdots + \zeta^{p-2}x_{p-1} &= 0,\\
&\vdots\\
x_0 + \zeta^{p-1} x_1 + \zeta^{p-2} x_2 + \cdots + \zeta x_{p-1} &= 0.
\end{aligned}$$

The action of $(0,1)$ fixes hyperplanes of the form

$$x_i = 0, \text{ for } 0 \leq i \leq p-1.$$

Note, that the fixed planes are each **set-wise** fixed, not **point-wise** fixed.
Finally,

$$\begin{aligned}(1,n)\cdot(x_0,\ldots,x_{p-1}) &= (1,0)\cdot(x_0,\zeta^n x_1,\zeta^{2n}x_2,\ldots,\zeta^{(p-1)n}x_{p-1})\\ &= (\zeta^{(p-1)n}x_{p-1},x_0,\zeta^n x_1,\ldots,\zeta^{(p-2)n}x_{p-2}).\end{aligned}$$

Note: this is equivalent to

$$\begin{aligned}(0,n)\cdot(x_{p-1},x_0,\ldots,x_{p-2}) = (x_{p-1},\zeta^n x_0,\zeta^{2n}x_1,\ldots\zeta^{(p-1)n}x_{p-2})\\ = \zeta^{(p-1)n} * (x_{p-1},\zeta^n x_0,\zeta^{2n}x_1,\ldots\zeta^{(p-1)n}x_{p-2}) = (\zeta^{(p-1)n}x_{p-1},x_0,\zeta^n x_1,\ldots,\zeta^{(p-2)n}x_{p-2}),\end{aligned}$$

using scalar invariance of projective points, verifying the group action is truly well-defined under composition.

This action has fixed planes

$$\begin{aligned}\zeta^n x_0 + x_1 + x_2 + \cdots x_{p-2} + x_{p-1} &= 0,\\ x_0 + \zeta^n x_1 + x_2 + \cdots x_{p-2} + x_{p-1} &= 0,\\ &\vdots\\ x_0 + x_1 + x_2 + \cdots x_{p-2} + \zeta^n x_{p-1} &= 0.\end{aligned}$$

Thus there is a correspondence between each point of $\mathbb{P}^1_{\mathbb{F}_p}$ and a set of p invariant planes of $\mathbb{P}^{p-1}_K$. Thus we can take $\mathcal{A}$ to be the total set of $p(p+1)$ invariant planes.

Now consider the transformation

$$\iota : \mathbb{P}^{p-1} \to \mathbb{P}^{p-1}$$

where $\iota = (1) \oplus J_{p-1}$, where J_{p-1} is the $(p-1)\times(p-1)$ exchange matrix. Viewing ι as a $p \times p$ matrix, we can see that

$$\mathcal{E}_1(\iota) = \langle e_0, e_i + e_{p-i+1} : 1 \leq i \leq (p-1)/2\rangle\,,$$

(assuming p is an odd prime) and

$$\mathcal{E}_{-1}(\iota) = \langle e_i - e_{p-i+1} : 1 \leq i \leq (p-1)/2\rangle,$$

so as vector spaces, $\dim \mathcal{E}_1 = (p+1)/2$ and $\dim \mathcal{E}_{-1} = (p-1)/2$. After projectivizing, this gives us F^+, a $(p-1)/2$-dimensional hyperplane, and F^-, a $(p-3)/2$-dimensional hyperplane.

The transforms of F^- under the group G defines a set $\mathcal{B}$ of p^2 hyperplanes of dimension $(p-3)/2$. Each of the hyperplanes of $\mathcal{B}$ is in $p+1$ of the hyperplanes of $\mathcal{A}$, and each hyperplane in $\mathcal{A}$ contains exactly p of the hyperplanes of $\mathcal{B}$. This gives us a geometric realization of a $(p(p+1)_p, p^2_{p+1})$-configuration.

2.3 Return to the Elliptic Embedding

Recall the linear system $|px_0|$ defines an embedding of the elliptic curve into $\mathbb{P}^{p-1}$. Then x_0 is in the hyperplane F^-, so the image of each G-translate of x_0 is mapped to F^-. Each hyperosculating point belongs to a unique hyperplane $g(F^-)$. A hyperplane from $\mathcal{A}$ cuts out E in p hyperosculating points. When $p = 3$, we get the **Hesse configuration** (or Wendepunkts-configuration).

2.4 Ceva Configuration

Let $n \in \mathbb{N}$ and μ_n be the group of n^{th} roots of unity of a field K of prime characteristic. Then we may define the n^3 points in $\mathbb{P}^2_K$ as follows:

$$P_{0,\zeta} = (0, 1, \zeta),\ P_{1,\zeta} = (\zeta, 0, 1),\ P_{2,\zeta} = (1, \zeta, 0),$$

where $\zeta \in \mu_n$. Let

$$\mathbf{L}_{0,\zeta} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \zeta \end{pmatrix},$$
$$\mathbf{L}_{1,\zeta} = \begin{pmatrix} 0 & 1 & 0 \\ \zeta & 0 & 1 \end{pmatrix},$$
$$\mathbf{L}_{2,\zeta} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & \zeta & 0 \end{pmatrix}.$$

Then the lines $L_{0,\alpha}, L_{1,\beta}, L_{2,\gamma}$ meet at one point $p_{\alpha,\beta,\gamma}$ if and only if $\alpha\beta\gamma = -1$. Thus we obtain n^2 points which together with the $3n$ lines $L_{i,\zeta}$ form an $(n^2_3, 3n_n)$-configuration. This is called the **Ceva configuration** and will be denoted Ceva(n).

Observation: I think the nine points in each fundamental plane of the Penrose configuration form a $(9_3, 9_3)$ Ceva configuration.

For $n \neq 3$, the symmetry group of Ceva(n) is $\mu_n^2 \rtimes S_3$; it is generated by permutations of $\mathbb{P}^2$-coordinates and homotheties $(x_0, x_1, x_2) \mapsto (\alpha x_0, \beta x_1, \gamma x_2)$.

When $n = 3$, the symmetry group is larger; for example, over $\mathbb{P}^2(\mathbb{F}_4)$, when we get an additional symmetry realized by the Frobenius automorphism, and a duality automorphism, so we get the Hesse group of order 216.

Blowing up the set of points $p_{\alpha,\beta,\gamma}$, we get a rational surface V together with a morphism $\pi : V \to \mathbb{P}^1$ whose general fibre is a nonsingular curve of genus $g = (n-1)(n-2)/2$. There are 3 singular fibres; each is the untion of n smooth rational curves with self-intersection $1 - n$ intersecting at one point. The morphism admits n^2 disjoint sections; each is a smooth rational curve with self-intersection -1. The Ceva configuration is realized by the set of sections and the set of irreducible components of singular fibres. If $n \neq 3$, the symmetry group of the configuration is realized by an automorphism group of the surface. There is a realization of Ceva(3) which realizes a subgroup of index 2 of Sym(Ceva(3)).

2.5 The Hesse-Salmon Configuration

The Hesse-Salmon Configuration is the general projection of the Reye $(12_4, 16_3)$-configuration into the plane: we get 12 points on a cubic curve E that are formed by taking a line L through E meeting at $L.E = A + B + C$, and taking $E[2] = \langle \gamma_1, \gamma_2 \rangle$ and forming $\{A, B, C\} + E[2]$.

2.6 Double-six

This is a (6_5)-configuration realized by a double-six of lines on a nonsigular cubic surface. The full symmetry group is the double extension of S_6. It is generated by permutation of lines in one family and a switch. In the full group of symmetries of 27 lines on a cubic surface this is the subgroup of $W(E_6)$ of index 36 which fixes a subset $\{\alpha, -\alpha\}$, where α is a positive root. (So $\#W(E_6) = 51840$. Wow!) One can realize the full symmetry group over a field of characteristic 2 by considering the Fermat cubic surface $x^3 + y^3 + z^3 + w^3 = 0$. Its automorphism group is isomorphic to the Weyl group $W(E_6)$.