The 27 lines on a smooth cubic surface C are the complete intersection of C and a nonic surface N.

Consider C as the blowup of $\mathbb{P}^2$ at six points $P_1, \ldots, P_6$. Then the 27 lines are given by the exceptional lines $e_1, \ldots, e_6$, the proper transforms of the lines $\ell - e_i - e_j$ $(i \neq j)$ and the proper transforms of the conics $2\ell - e_1 - \cdots - e_6 + e_i$ $(1 \leq i \leq 6)$. Then the sum of the 27 lines is the divisor

$$27\ell - 9e_1 - 9e_2 - 9e_3 - 9e_4 - 9e_5 - 9e_6 = 9(-K_C)$$

where K_C is the canonical divisor of C. Using the adjunction formula, we know

$$K_C = (K_{\mathbb{P}^3} + C)|_C = (-4H + 3H)|_C = -H|_C,$$

and so $-K_C = H|_C$ is the class of a plane intersecting C. Therefore the sum of the 27 lines is $9(H|_C) = (9H)|_C$, the class of a nonic intersecting C.

This requires knowledge that the blowup of $\mathbb{P}^2$ at six points is cubic. Without knowing this, one can use the generators $(f_1, \ldots, f_4) = H^0(3\ell - P_1 - \cdots - P_6)$. This map sends $P \mapsto (f_1(P), f_2(P), f_3(P), f_4(P)) \in \mathbb{P}^3$, so $\dim|-K_C| = 3 = \dim|H \in \mathrm{Cl}\mathbb{P}^3|$ does that work?

You can also just prove that $\mathrm{Bl}_{P_1+\cdots+P_6}(\mathbb{P}^2)$ is cubic by the example

$$\{P_1, \ldots, P_6\} = \{(1,0,0), (0,1,0), (0,0,1), (1,-1,0), (1,0,-1), (0,1,-1)\} = Z.$$

In this case $H^0(3\ell - Z) = (xyz, xy(x+y+z), xz(x+y+z), yz(x+y+z))$. So we can build a rational map

$$F : \mathbb{P}^2 \dashrightarrow \mathbb{P}^3$$

given by

$$F(a,b,c) = (abc, ab(a+b+c), ac(a+b+c), bc(a+b+c))$$

where $\mathbb{P}^3$ has coordinates ξ, υ, ζ, ω. The image of F is the cubic surface

$$V\left(\upsilon\zeta\omega - \xi(\upsilon\zeta + \upsilon\omega + \zeta\omega)\right).$$

Furthermore, F is injective as we shall see. Suppose there were two distinct points $P \neq Q \in \mathbb{P}^2$ such that $F(P) = F(Q)$. Then

$$\begin{aligned}&(f_1(P), f_2(P), f_3(P), f_4(P))\\ =&(f_1(Q), f_2(Q), f_3(Q), f_4(Q))\end{aligned}$$

and so every 2×2 minor of the matrix $\begin{pmatrix} f_1(P) & f_2(P) & f_3(P) & f_4(P) \\ f_1(Q) & f_2(Q) & f_3(Q) & f_4(Q) \end{pmatrix}$ is 0. Therefore $f_i(P)f_j(Q) - f_j(P)f_i(Q) = 0$ for all i, j. Now let us find $H^0(3\ell - Z - P)$. This linear system is generated by the three polynomials

$$\begin{aligned} g_1 &= f_1(x,y,z)f_2(P) - f_1(P)f_2(x,y,z) \\ g_2 &= f_1(x,y,z)f_3(P) - f_1(P)f_3(x,y,z) \\ g_3 &= f_1(x,y,z)f_4(P) - f_1(P)f_4(x,y,z). \end{aligned}$$

But then $g_1(Q) = g_2(Q) = g_3(Q) = 0$, and so any cubic that contains Z and P must also contain Q, so the eight points do not impose independent conditions. This shouldn't happen, so such a pair P and Q do not exist. Thus F is injective.

Therefore $F(\mathbb{P}^2)$ is truly a rational cubic surface in $\mathbb{P}^3$.

We can blow up the plane $\mathbb{P}^2$ at a point $P = (0, 0, 1)$ and get an embedding

$$F : \mathbb{P}^2 \dashrightarrow \mathbb{P}^4$$

via

$$(a, b, c) \mapsto (a^2, ab, ac, b^2, bc)$$

and getting the surface

$$F(\mathbb{P}^2) \cong V(yz - xu, zw - yu, xw - y^2).$$

Then for $(t, u) \in \mathbb{P}^1$, $F(at, bt, u) = (a^2t^2, abt^2, atu, b^2t^2, btu) = (a^2t, abt, au, b^2t, bu)$ and so for $t = 0$ we get an output of $(0, 0, a, 0, b)$.