Let X be a curve of genus $g \geq 2$ and let K be the canonical divisor on X. A *gap number* for a point $p \in X$ is a number $n \geq 0$ such that $\ell(K - np) \neq \ell(K - (n-1)p)$. By Riemann-Roch, this holds if and only if $\ell(np) = \ell((n-1)p)$. Denote the sequence of gap numbers for a point $p \in X$ as $G_p(K) = \{1 = n_1 < n_2 < \cdots < n_\alpha\}$. We shall show that $\alpha = g$.

By Riemann-Roch, we know that $\ell(np) = n - g + 1$ for $n \geq 2g - 1$. Thus $\ell((2g-1)p) = g$ and the dimension increases by 1 for each subsequent increment. Therefore for the first $2g-2$ entries of the $\ell(np)$ sequence, there will be $g-1$ spots where the dimension increases by 1, and $g-1$ spots where the dimension does not increase. Thus for each point $p \in X$, we know $\#G_p = g$, as 1 is always the first gap number of any point.

A point $p \in X$ is a *Weierstrass point* of X if its gap sequence is anything other than $\{1, 2, \ldots, g\}$. In other words, if $\ell(gp) > 1$ or $\ell(K - gp) > 0$. The *weight* of a point is given by

$$w_p(K) = \sum_{i=1}^{g} (n_i - i)$$

where n_i is the i^{th} member of the gap sequence $G_p(K)$. It is easy to check that the weight of a non-Weierstrass point is 0.

Miranda proves in Theorem 4.15 that if Q is a g^r_d and $w_p(Q)$ is defined similarly using the gap numbers n_i such that $\ell(Q - n_i p) \neq \ell(Q - (n_i - 1)p)$, then

$$\sum_{p \in X} w_p(Q) = (r+1)(d + rg - r).$$

Since K is a g^{g-1}_{2g-2}, we have that the weights of the Weierstrass points add up to $g(g^2 - 1)$. Thus there are finitely many Weierstrass points.

Note that the set of non-gap numbers $\mathbb{N} \backslash G_p(K)$ forms a semigroup under addition. If n, m are two non-gap numbers, then $\ell(np) > \ell((n-1)p)$ and $\ell(mp) > \ell((m-1)p)$ and so there are meromorphic functions f and g having poles at p of order n and m respectively, and no poles anywhere else. Then fg is a meromorphic function with a pole of order $n + m$ at p and no poles anywhere else. Thus $fg \in \mathcal{L}((n+m)p) \backslash \mathcal{L}((n+m-1)p)$ so $\ell((n+m)p) > \ell((n+m-1)p)$ thus $n + m$ is a non-gap number of p.

Next we shall show that there is an upper bound on the weight of a Weierstrass point of $g(g-1)/2$, which is only attained if X is hyperelliptic. We shall follow the proof provided by Shor and Shaska *Weierstrass points of superelliptic curves.*

We shall begin by looking at the non-gap sequence $N_p(K)$ in $\{1, 2, \ldots, 2g\}$. That is, $N_p(K) = [2g] \setminus G_p(K) = \{\alpha_1, \ldots, \alpha_g\}$ where $1 < \alpha_1 \cdots < \alpha_g = 2g$. Then for all $1 \leq j < g$, we will show $\alpha_j + \alpha_{g-j} \geq 2g$.

Suppose there is a $j < g$ such that $\alpha_j + \alpha_{g-j} < 2g$. Then for all $k \leq j$, we know $\alpha_k + \alpha_{g-j} < 2g$. And since $N_p(K)$ is contained in a semigroup under addition, we know that

$\alpha_k + \alpha_{g-j} \in N_p(K)$ with $\alpha_{g-j} < \alpha_k + \alpha_{g-j} < \alpha_g = 2g$. Since this holds for all $1 \le k \le j$, there would be j of these non-gaps. However, only $j-1$ non-gaps exists between α_{g-j} and α_g. This is a contradiction. Thus no such j can exist.

Now we shall show that for all $p \in X$, that $w_p(K) \le g(g-1)/2$, and that equality holds if and only if X is hyperelliptic. Recall that

$$\begin{aligned} w_p(K) = \sum_{i=1}^{g}(n_i - i) = \sum_{i=1}^{g} n_i - \sum_{i=1}^{g} i = \sum_{i=1}^{2g} i - \sum_{i=1}^{g} \alpha_i - \sum_{i=1}^{g} i \\ = \sum_{i=g+1}^{2g} i - \sum_{i=1}^{g} \alpha_i = \sum_{i=g+1}^{2g-1} i - \sum_{i=1}^{g-1} \alpha_i \end{aligned}$$

since $\alpha_g = 2g$. The first sum is $3g(g-1)/2$. The second sum is at least $(g-1)/2$ many summands of at least $2g$, and so $\sum_{i=1}^{g-1} \alpha_i \ge g(g-1)$. Thus $w_p(K) \le 3g(g-1)/2 - g(g-1) = g(g-1)/2$.

The weight is maximized when the α_i values are minimized. This occurs when $\alpha_1 = 2$ and (since $N_p(K)$ is contained in a semigroup under addition), we get $N_p(K) = \{2, 4, 6, \dots, 2g\}$. Thus $G_p(K) = \{1, 3, 5, \dots, 2g-1\}$. Thus the sequence $\{\ell(np)\}$ looks like

$$\ell(0p) = 1,\ \ell(1p) = 1,\ \ell(2p) = 2,\ \ell(3p) = 2,\ \ell(4p) = 3,\ \ell(5p) = 3, \dots$$

or $\ell(np) = \lfloor \frac{n}{2} + 1 \rfloor$. Thus any meromorphic function with a pole only at p has an even-order pole at p. This non-constant function in $\mathcal{L}(2p)$ corresponds with a degree-2 map from X to $\mathbb{P}^1$ sending p (and only p) to ∞. This corresponds with the hyperelliptic map $X \to \mathbb{P}^1$, of which p must be a ramification point. Thus the maximum weight is only achieved when X is hyperelliptic.

Now recall that the sum of the weights of the Weierstrass points is $g(g^2-1)$ and each weight is at most $g(g-1)/2$, with equality if and only if X is hyperelliptic. Then there are at least $g(g^2-1)/(g(g-1)/2) = 2g+2$ Weierstrass points on X, with equality if and only if X is hyperelliptic.

Thus if X is not hyperelliptic, there are more than $2g+2$ Weierstrass points. Since they are determined by the canonical divisor K, any automorphism on X must permute the Weierstrass points. Furthermore, Corollary 2.10 in Miranda says that any nontrivial automorphism on a non-hyperelliptic algebraic curve of genus g has at most $2g+2$ fixed points. Since there are more than $2g+2$ Weierstrass points on X, any automorphism that fixes them all must be the identity. Thus $\mathrm{Aut}X$ is finite.

Inflection Point: If Q is a nonempty g^r_d, then $p \in X$ is an *inflection point* for the linear system Q if $G_p(Q) \neq \{1, 2, \ldots, r+1\}$.

Q is a linear system, $Q \subseteq |D|$ a complete linear system $V \subseteq \mathcal{L}(D)$ is the nonzero vector space corresponding to Q.

Wronskian: Given functions $g_1, \ldots, g_{r+1}$ of variable z, the *Wronskian* of $g_1, \ldots, g_{r+1}$ is

$$W_z(g_1, \ldots, g_r)(z) = \det \begin{pmatrix} g_1(z) & g_1'(z) & g_1^{(2)}(z) & \cdots & g_1^{(r)}(z) \\ g_2(z) & g_2'(z) & g_2^{(2)}(z) & \cdots & g_2^{(r)}(z) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ g_{r+1}(z) & g_{r+1}'(z) & g_{r+1}^{(2)}(z) & \cdots & g_{r+1}^{(r)}(z) \end{pmatrix}.$$

The Wronskian is holomorphic if each of the g_i's are holomorphic.

Lemma 4.4. If $g_1, \ldots, g_{r+1}$ are linearly independent holomorphic functions defined in a neighborhood of $z = 0$, then the Wronskian is not identically zero near $z = 0$.

Corollary 4.5. For a fixed linear system Q on an algebraic curve X, there are only a finite number of inflection points.

Proof. Fix a point $p \in X$; then there is a neighborhood U of p such that for all $q \in U$, we have $D(q) = 0$ if $q \neq p$. Fix a basis $\{g_1, \ldots, g_{r+1}\}$ for V. By the above analysis, we have that q is an inflectino point for Q if and only if the Wronskian is zero at q. Since the g_i's are linearly independent, this Wronskian is not identically zero; and since it is holomorphic, it has discrete zeros. Hence after shrinking U there will be no inflection points in $U \setminus \{p\}$. □

Definition 4.6. A *meromorphic n-fold differential* on an open set $V \subseteq \mathbb{C}$ is an expression μ of the form

$$\mu = f(z)(\mathsf{d}z)^n,$$

where f is a meromorphic function on V. We say that μ is a meromorphic n-fold differential *in the coordinate z*.

Lemma 4.9. Let X be an algebraic curve, and let $g_1, \ldots, g_\ell$ be meromorphic functions on X. Then

$$W_z(g_1(z), \ldots, g_\ell(z))(\mathsf{d}z)^{\ell(\ell-1)/2}$$

defines a meromoprhic $\ell(\ell-1)/2$-fold differential on X.

Lemma 4.10. Let X be an algebraic curve, D a divisor on X and let $f_1, \ldots, f_\ell$ be meromorphic functions in $\mathcal{L}(D)$. Then the meromorphic n-fold differential $W(f_1, \ldots, f_\ell)$ has poles bounded by ℓD:

$$W(f_1, \ldots, f_\ell) \in \mathcal{L}^{(\ell(\ell-2)/2)}(\ell D).$$

If one changes the basis of V, then the Wronskian changes by the determinant of the chage of basis matrix, and so the Wronskian is well defined (up to scalar constant) by the linear system Q itself, and not by the choice of basis. We will therefore denote it by $W(Q)$ when convenient; Lemma 4.10 implies that

$$W(Q) = \mathcal{L}^{(r(r+1)/2)}((r+1)D)$$

if $r = \dim Q$.

Lemma 4.11. Let X be an algebraic curve, D a divisor on X and $K = \operatorname{div}(\omega)$ a canonical divisor. Then the multiplication map

$$\zeta : \mathcal{L}(D + nK) \to \mathcal{L}^{(n)}(D)$$

defined by $\zeta(f) = f\omega^n$ is an isomorphism of vector spaces.

Proof. Since ω is a meromorphic 1-form, and f is a meromorphic function, then $f\omega^n$ is a meromorphic n-fold differential. Moreover the multiplication map is clearly linear in f, and is injective.

To show that $f\omega^n$ has poles bounded by D, fix a point $p \in X$ and a local coordinate z at p, and write $\omega = g(z)\mathsf{d}z$. Then $\omega^n = g(z)^n(\mathsf{d}z)^n$, so that

$$\operatorname{ord}_p(f\omega^n) = \operatorname{ord}_p(f) + n\operatorname{ord}_p(g) = \operatorname{ord}_p(f) + nK(p) \geq -D(p)$$

if $f \in \mathcal{L}(D + nK)$; hence we see that ζ does map $\mathcal{L}(D + nK)$ to $\mathcal{L}^{(n)}(D)$.

Finally to see that ζ is surjective, we note that if $\mu = h(z)(\mathsf{d}z)^n \in \mathcal{L}^{(n)}(D)$, and $\omega = g(z)\mathsf{d}z$, then $f = h/g^n$ is a meromorphic function in $\mathcal{L}(D+nK)$, which is defined globally. □

Corollary 4.12. Let X be an algebraic curve and Q a linear system on X with $r = \dim Q$. Then $\deg(\operatorname{div}(W(Q)) = \sum_{p\in X} \operatorname{ord}_p(W(Q)) = r(r + 1)(g - 1)$.

Proof. Let $n = r(r + 1)/2$, so that by Lemma 4.10 we have that the Wronskian differential $W(Q)$ is an element of the space $\mathcal{L}^{(n)}((r+1)D)$. Then by Lemma 4.11 there is a meromorphic 1-form ω and a meromorphic function f such that $W(Q) = f\omega^n$. Then

$$\begin{aligned}\sum_p \operatorname{ord}_p(W(Q)) &= \sum_p \operatorname{ord}_p(f\omega^n)\\ &= \sum_p [\operatorname{ord}_p(f) + n\operatorname{ord}_p(\omega)]\\ &= n\sum_p \operatorname{ord}_p(\omega) \text{ (since } \sum_p \operatorname{ord}_p(f) = 0)\\ &= n(2g - 2) = r(r + 1)(g - 1)\end{aligned}$$

using the fact that $\deg(\operatorname{div}(\omega)) = 2g - 2$. □

Note that by Lemma 4.3, p is an inflection point for $|D|$ if and only if the Wronskian $W_z(z^{D(p)}f_1, \ldots, z^{D(p)}f_{r+1})$ is zero at p. Since

$$\operatorname{ord}_p(W_z(z^{D(p)}f_1, \ldots, z^{D(p)}f_{r+1})) = \operatorname{ord}_p(z^{(r+1)D(p)}W_z(f_1, \ldots, f_{r+1})) \overset{(4.13)}{=} (r+1)D(p)+\operatorname{ord}_p(W(Q))$$

we have a clean link between the order of vanishing of $W_z(z^{D(p)}f_1, \ldots, z^{D(p)}f_{r+1})$ (which measures inflectionary behavior) and $\operatorname{ord}_p(W(Q))$, for which we have a global formula.

Lemma 4.14. If $G_p(Q) = \{n_1 < n_2 < \cdots < n_{r+1}\}$ and $\{f_1, \ldots, f_{r+1}\}$ is a basis for V, then

$$\operatorname{ord}_p(W_z(z^{D(p)}f_1, \ldots, z^{D(p)}f_{r+1})) = \sum_{i=1}^{r+1}(n_i - i).$$

Theorem 4.15. Let X be an algebraic curve of genus g, and let Q be a g_d^r on X. Then

$$\sum_{p\in X} w_p(Q) = (r+1)(d+rg-r).$$

Proof. Choose a basis $f_1, \ldots, f_{r+1}$ for the subspace V of $\mathcal{L}(D)$ corresponding to Q. We compute

$$\begin{aligned}\sum_p w_p(Q) &= \sum_p \mathrm{ord}_p(W_z(z^{D(p)}f_1, \ldots, z^{D(p)}f_{r+1})) \text{ (4.14)}\\ &= \sum_p [(r+1)D(p) + \mathrm{ord}_p(W(Q))] \text{ (4.13)}\\ &= (r+1)d + r(r+1)(g-1) = (r+1)(d+rg-r)\end{aligned}$$

using Corollary 4.12. □

Corollary 2.9. Let X be a nonhyperelliptic curve of genus g.

(a) For a general positive divisor D of degree $d \leq g$, $\dim \mathcal{L}(D) = 1$ and $\dim H^1(D) = g-d$, so that $\dim |D| = 0$ and $|D| = \{D\}$.

(b) For a general positive divisor D of degree $d \geq g$, $H^1(D) = 0$ and $\dim \mathcal{L}(D) = d+1-g$, so that $\dim |D| = d - g$.

Corollary 2.10. Suppose that X is a nonhyperelliptic algebraic curve of genus g. Then any nontrivial automorphism of X has at most $2g+2$ fixed points.

Proof. By Corollary 2.9, we may choose $g+1$ general points $p_1, \ldots, p_{g+1}$ on X and find a meromorphic function f on X with simple poles at each p_i and no other poles. If $\sigma \in \mathrm{Aut} X$ and is not the identity, then $g = f - f \circ \sigma$ has at most $2g+2$ poles, namely the p_i's and the $\sigma^{-1}(p_i)$'s. Therefore g has at most $2g+2$ zeros also. But any point fixed by σ is a zero of g. $\square$