Let $C = V(y^2z - x^3 - x^2z)$ be a nodal cubic with node $n = (0 : 0 : 1)$ over a field k. Let L be the line $V(x - z)$. Then the two tangent lines to C at N meet L at $z = (1 : -1 : 1)$ and $i = (1 : 1 : 1)$. Then $L \setminus \{z, i\} \cong k^\times$ as a group, via the bijection $f : L \to \mathbb{P}^1$ defined by $f(a : b : a) = (a + b : b - a)$. Note then that $f(i) = (1 : 0)$ and $f(z) = (0 : 1)$. For $(a : b), (c : d) \in \mathbb{P}^1 \setminus \{(1 : 0), (0 : 1)\}$, multiplication is defined as

$$(a : b) \cdot (c : d) = (ac : bd).$$

This way, $\mathbb{P}^1_k \setminus \{(1 : 0), (0 : 1)\} \cong k^\times$ via $(a : b) \mapsto a/b$.

We can define multiplication $\cdot_L$ on $L \setminus \{i, z\}$ by

$$(a : b : a) \cdot_L (c : d : c) = f^{-1}(f(a : b : a) \cdot f(c : d : c)) = (ad + bc : ac + bd : ad + bc).$$

We can also define a projection map $\pi : L \to C$ via the node n, where

$$\pi(a : b : a) = (b^2a - a^3 : b^3 - ba^2 : a^3).$$

This is because the line that connects $(a : b : a)$ with $n = (0 : 0 : 1)$ is $V(bx - ay)$. Subbing in $y = bx/a$, we get $(bx/a)^2z - x^3 - x^2z = 0$, or $b^2x^2z - a^2x^3 - a^2x^2z = 0$. When $x, z \neq 0$, we have $a^2x + (a^2 - b^2) = 0$, or $x = (b^2 - a^2)/a^2$. Since $y = bx/a$, we have $y = (b^3 - ba^2)/a^3$ and $z = 1$. So the point is $((b^2 - a^2)/a^2 : (b^3 - ba^2)/a^3 : 1) = (b^2a - a^3 : b^3 - ba^2 : a^3)$.

We can check that we in fact have

$$(b^3 - ba^2)^2a^3 - (b^2a - a^3)^3 - (b^2a - a^3)^2a^3 = 0.$$

To show that $C \cong k^\times$, we want to show that

$$\pi^{-1}(A + B) = \pi^{-1}(A) \cdot_L \pi^{-1}(B).$$

Let $\pi^{-1}(A) = (a : b : a)$, so $A = (b^2a - a^3 : b^3 - ba^2 : a^3)$ and $\pi^{-1}(B) = (c : d : c)$, so $B = (d^2c - c^3 : d^3 - dc^2 : c^3)$. Then

$$\pi^{-1}(A) \cdot_L \pi^{-1}(B) = (a : b : a) \cdot_L (c : d : c) = (ad + bc : ac + bd : ad + bc).$$

Then

$$\begin{aligned}\pi(\pi^{-1}(A) \cdot_L \pi^{-1}(B)) &= \pi(ad + bc : ac + bd : ad + bc) \\ &= ((ac + bd)^2(ad + bc) - (ad + bc)^3 : (ac + bd)^3 - (ac + bd)(ad + bc)^2 : (ad + bc)^3).\end{aligned}$$

In order to show $\pi(\pi^{-1}(A) \cdot_L \pi^{-1}(B)) = A + B$, we need to determine whether A, B, and $-\pi(\pi^{-1}(A) \cdot_L \pi^{-1}(B))$ are collinear. Note

$$-\pi(\pi^{-1}(A) \cdot_L \pi^{-1}(B)) = ((ac+bd)^2(ad+bc) - (ad+bc)^3 : -(ac+bd)^3 + (ac+bd)(ad+bc)^2 : (ad+bc)^3).$$

So we can find that the determinant of the 3×3 matrix

$$\begin{pmatrix} (ac + bd)^2 (ad + bc) - (ad + bc)^3 & (ac + bd) (ad + bc)^2 - (ac + bd)^3 & (ad + bc)^3 \\ b^2a - a^3 & b^3 - ba^2 & a^3 \\ d^2c - c^3 & d^3 - dc^2 & c^3 \end{pmatrix}$$

is indeed 0, thanks to an online calculator. So we indeed have

$$\pi(\pi^{-1}(A) \cdot_L \pi^{-1}(B)) = A + B.$$

Thus

$$\pi^{-1}(A) \cdot_L \pi^{-1}(B) = \pi^{-1}(A + B),$$

and so $\pi^{-1} : C \setminus \{n\} \to L \setminus \{i, z\} \cong k^\times$ is indeed a group isomorphism.