Let E be the elliptic curve given by the polynomial $y^2z - x^3 + xz^2$. Let $P = (0, 0, 1) \in E$. We will consider the the generators of $L(P)$, 1 and y/x. Note that $\mathrm{ord}_P(y/x) = 1 - 2 = -1$.

We will consider the map $f : E \to \mathbb{P}^1$ determined by the linear system $|P|$, given by $f(Q) = (1, y(Q)/x(Q))$. Note that f is 2-to-1 because the line L_α given by $y/x = \alpha$ intersects E at two points different from P. Given $L_\alpha.E = A_\alpha + B_\alpha + P$, then $f(A_\alpha) = f(B_\alpha)$.

For α such that $A_\alpha = B_\alpha$, then A_α is a ramification point, and $f(A_\alpha)$ is a branch point. This occurs when L_α is tangent to E at A_α, and therefore $A_\alpha \in \mathcal{P}_P(E) \cap E$. Since $\#(\mathcal{P}_P(E) \cap E) = 4$, we have that there are 4 ramification points (as also predicted by the Riemann-Hurwitz formula).

Note that $f(P) = (1, \infty)$, since $\mathrm{ord}_P(y/x) = -1$, so we have $f(P) = (0, 1)$. Also $f(0, 1, 0) = (0, 1)$. We have $f(-1, 0, 1) = (1, 0)$, and $f(1, 0, 1) = (1, 0)$.

A claim I found in a paper about theta characteristics: If X is a smooth curve and there exist distinct points $P, Q, R, S \in X$ we have $P + Q \sim R + S$, then X is hyperelliptic. (We will just assume X is quartic.)

Suppose $P + Q \sim R + S$, then $P + Q - R - S = \mathrm{div}(f)$ and f yields a hyperelliptic map $X \to \mathbb{P}^1$. We know f is a rational function such that $\mathrm{ord}_P(f) = \mathrm{ord}_Q(f) = 1$ and $\mathrm{ord}_R(f) = \mathrm{ord}_S(f) = -1$ and $\mathrm{ord}_A(f) = 0$ for all $A \in X \setminus \{P, Q, R, S\}$. Then we must have a line L_1 connecting P and Q meeting the line L_2 connecting R and S at the same points: that is, $L_1.X = P + Q + A + B$ and $L_2.X = R + S + A + B$. Since P, Q, R, S are distinct, we must have $A = B$. In fact, we force A to be a singularity of multiplicity 2 on X. Therefore f determines a degree-2 map $X \to \mathbb{P}^1$ determined by lines through A, the same way the above map was determined by lines L_α through $P \in E$. Thus X is hyperelliptic.

When X is smooth $g \geq 3$, the canonical divisor K determines an embedding. For example, with $\deg X = 4$, $K_X = L.X = A + B + C + D$, four collinear points on X. Note that $\mathcal{O}_{\mathbb{P}^2}(L)$ is three dimensional, spanned by x, y, and z. Therefore $\ell(K) = h^0(K) = \dim |K| + 1 = 3$, and the linear system $|K|$ defines the map $k : X \to \mathbb{P}^2$ given by $k(P) = (x(P), y(P), z(P))$.

With $\deg X = 5$, we have $K_X = 2L.X =$ ten coconical points. Since $\mathcal{O}_{\mathbb{P}^2}(2L) = 6$, generated by x^2, xy, xz, y^2, yz, and z^2, we get the embedding $X \to \mathbb{P}^5$ given by $P \mapsto (x^2(P), xy(P), xz(P), y^2(P), yz(P), z^2(P))$.

In general when you have a divisor D you can determine a linear system by taking the generators $f_1, \ldots, f_n$ of $\mathcal{L}(D)$ and sending P to the point $(f_1(P), \ldots, f_n(P)) \in \mathbb{P}^{\ell(D)-1}$.