Let $\mathbb{P}^3_q := \mathbb{P}^3_{\mathbb{F}_q}$. For $a, b \in \mathbb{F}_{q^2}$, denote by $[a, b]$ the set $\{(x, ax + bx^q) : x \in \mathbb{F}_{q^2}\} \subseteq \mathbb{F}_{q^2} \times \mathbb{F}_{q^2}$. This is a line in $\mathbb{F}^2_{q^2}$, which corresponds to a plane in $\mathbb{F}^4_q \cong_{\mathbb{F}_q} \mathbb{F}^2_{q^2}$ and so corresponds to a line in $\mathbb{P}^3_q$. For example, choosing $q = 3$ and $\mathbb{F}_9 = \mathbb{F}_3[i]/(i^2+1)$, $a = 1+i \in \mathbb{F}_9$ and $b = 2+0i \in \mathbb{F}_9$ and the $\mathbb{F}_3$-vector space isomorphism $f : \mathbb{F}_9 \to \mathbb{F}^2_3$ via $f(\mu + \lambda i) = (\mu, \lambda)$, we get

$$[a, b] = \{\vec{0}, (1 + i, 2), (2i, 1 + i), (1 + 2i, 1 + 2i), (2, 2i), (2 + 2i, 1), (i, 2 + 2i), (2 + i, 2 + i), (1, i)\}$$

in $\mathbb{F}^2_9$ which via f corresponds to the plane in $\mathbb{F}^4_3$

$$\{\vec{0}, (1, 1, 2, 0), (0, 2, 1, 1), (1, 2, 1, 2), (2, 0, 0, 2), (2, 2, 1, 0), (0, 1, 2, 2), (2, 1, 2, 1), (1, 0, 0, 1)\}$$

which in $\mathbb{P}^3_3$ corresponds to the line

$$\{(1, 1, 2, 0), (0, 1, 2, 2), (1, 2, 1, 2), (1, 0, 0, 1)\}.$$

Bruen in *Partial Spreads and Replaceable Nets* describes a recipe for creating maximal partial spreads of size $q^2 - q + 2$ (Theorem 3.6). We will follow this recipe for $q = 11$. First we will take $\mathbb{F}_{11} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, T\}$, defining $T = 10$ for simplicity. We will look at $\mathbb{F}_{121} = \mathbb{F}_{11}[\alpha]/(\alpha^2 + 1)$, so $\alpha^2 = T$. Note that $[a + b\alpha, 0]$ is the line that connects $(1, 0, a, b)$ with $(0, 1, -b, a)$.

First we will take 12 lines in a regulus R: our regulus will specifically be made up of the lines $r_a := [a, 0]$ for $a \in \mathbb{F}_{11}$ and ℓ_∞, the line through $(0, 0, 1, 0)$ and $(0, 0, 0, 1)$. Choose two lines in R to denote u and v: we will choose $u = r_0$ and $v = \ell_\infty$. For our regulus R, there is a cross-regulus $R^\times$ comprising 12 mutually-skew lines, with each line of $R^\times$ intersecting each line of R once: therefore giving R and $R^\times$ the same set of 121 points in $\mathbb{P}^3_{11}$. We will select two lines from $R^\times$: u' the line through $(0, 0, 1, 0) \in \ell_\infty$ and $(1, 0, 0, 0) \in r_0$; and v' the line through $(0, 0, 0, 1) \in \ell_\infty$ and $(0, 1, 0, 0) \in r_0$.

We will then pick $P = (1, 0, 0, 0) \in u' \cap v$, $Q = (0, 0, 0, 1) \in v' \cap u$, $A = (0, 0, 1, 0) \in u' \cap u$, and $B = (0, 1, 0, 0) \in v' \cap v$. Then define $\ell = \overline{AB}$ and $m = \overline{PQ}$. Pick a point $x \in \ell$ not in the regulus: we will choose $x = (0, 1, 1, 0)$. Note $\ell \cap m = \varnothing$. What we want is a line meeting ℓ and m and is skew to all lines of R. From x, we will draw 10 lines to m (one to each point of m not in R) and name them t_a if the line intersects r_a. We will have $t_1 = t_T$, $t_5 = t_6$, $t_2 = t_9$, $t_4 = t_7$, and $t_3 = t_8$. This leaves five lines from x to m that are skew to every line in R. Pick one of those lines and call it y. We will choose $y = V(x + w, y - z)$. Then R and y will determine a complete spread S.

Note that since y contains the points $(0, 1, 1, 0)$ and $(1, 0, 0, T)$, we can determine if it is a line of the form $[a+b\alpha, c+d\alpha]$ for some $a+b\alpha, c+d\alpha \in \mathbb{F}_{121}$. Take $(1, 0, a+c, b+d) = (1, 0, 0, T)$ and $(0, 1, d - b, a - c) = (0, 1, 1, 0)$. Then $a + c = 0$, $b + d = T$, $d - b = 1$, and $a - c = 0$. Solving this system of equations yields $a = c = 0$ and $b = T$ and $d = 0$. So we in fact get $y = [T\alpha, 0]$. Thus the complete spread determined by $R + y$ is in fact the "canonical" spread $S = \{[a, 0] : a \in \mathbb{F}_{121}\} \cup \{\ell_\infty\}$.

Denote by G the regulus determined by u, v, and y. Note $\ell \in G^\times$. In fact, we can write $G = \{[a\alpha, 0] : a \in \mathbb{F}_{11}\} \cup \{\ell_\infty\} \subseteq S$.

We will then write $\mathscr{B} = S \setminus (R \cup G)$, where R is the $[a, 0]_\infty$ regulus and G is the $[a\alpha, 0]_\infty$ regulus. Note that $R \cap G = \{r_0, \ell_\infty\}$. Denote by $G^* = G \setminus \{r_0, \ell_\infty\}$. Then $S_1 = \mathscr{B} \cup G^* \cup R^\times$ is a spread.

Now, recall $\ell = \overline{AB}$. Denote by $\mathscr{A}$ the lines of S_1 that meet ℓ; $\mathscr{A}$ is not a regulus (despite $\#\mathscr{A} = 12$); $\mathscr{A}$ has two transversals: ℓ and m. Finally, denote by W the set $S_1 \setminus \mathscr{A} \cup \{\ell, m\}$. W is a maximal partial spread of size $q^2 - q + 1$ (Theorem 3.5 in Bruen). In our $q = 11$ case, that is 111.

Set theoretically, we have

$$W = (((S \setminus (R \cap G)) \cup G^* \cup R^\times) \setminus \mathscr{A}) \cup \{\ell, m\}.$$

So $\mathbb{P}^3_{11} \setminus W$ is the set of points on lines of S_1 that meet ℓ other than points actually on ℓ or m. So the complement has $q^3+q^2+q+1-(q^2-q+2)(q+1) = q^3+q^2+q+1-(q^3+q+2) = q^2-1$ points.

Note that

$$\mathscr{A} = \{u', v', y = r_{T\alpha}, r_{9\alpha}, r_{8\alpha}, \ldots, r_\alpha\}$$

where $u', v' \in R^\times$ and $r_{T\alpha}, r_{9\alpha}, r_{8\alpha}, \ldots, r_\alpha \in G^*$. In our case, we have $G^* \subseteq \mathscr{A}$ (I guess in Bruen's general construction, this isn't necessarily true?) and so we can simplify the construction of W considerably. We get $W = (S \setminus (R \cap G)) \cup (R^\times \setminus \{u', v'\}) \cup \{\ell, m\}$.