Proposition 1. Let A be an $n \times n$ Markov matrix. Then A has an eigenvalue of 1.

Proof. To show that A has an eigenvalue of 1, we need to show that $\det(A - I_n) = 0$. Since A is a Markov matrix, we know that the entries of each column add up to 1. Let

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{bmatrix}$$

where

$$\sum_{i=1}^{n} a_{i,j} = 1$$

for all j (i.e. each column's entries add to 1). Then

$$A - I_n = \begin{bmatrix} a_{1,1} - 1 & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} - 1 & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} - 1 \end{bmatrix}.$$

Now the sum of all the entries of column j of $A - I_n$ is

$$\left(\sum_{i=1}^{n} a_{i,j} \right) - 1 = 1 - 1 = 0,$$

so all the entries of the columns of $A - I_n$ add up to 0. So if we consider each column of $A - I_n$ as an individual vector,

$$\vec{v}_1 = \begin{bmatrix} a_{1,1} - 1 \\ a_{2,1} \\ \vdots \\ a_{n,1} \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} a_{1,2} \\ a_{2,2} - 1 \\ \vdots \\ a_{n,2} \end{bmatrix}, \cdots, \vec{v}_n = \begin{bmatrix} a_{1,n} \\ a_{2,n} \\ \vdots \\ a_{n,n} - 1 \end{bmatrix},$$

we see that all the entries of the vectors $\vec{v}_j$ satisfy the homogeneous linear equation

$$x_1 + x_2 + \cdots + x_n = 0$$

because all the entries in each column add up to 0. As we learned in the very very beginning of the semester, the solution sets of linear equations are $n-1$-dimensional hyperplanes in $\mathbb{R}^n$. And since this linear equation is homogenous, its corresponding hyperplane contains the origin, so it is a vector space! Specifically, the hyperplane is the nullspace of the matrix

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix},$$

which yields 1 basic variable and $n - 1$ free variables.

So this means that the n vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are all in an $n-1$-dimensional vector space together. Therefore $\dim \operatorname{span}\{\vec{v}_1, \ldots, \vec{v}_n\} \leq n - 1 < n$, so the vectors $\vec{v}_1, \ldots, \vec{v}_n$ are linearly dependent. Since the $\vec{v}_j$ vectors are the columns of $A - I_n$, this means that $\operatorname{rank}(A - I_n) < n$, so $A - I_n$ is not surjective, so $A - I_n$ is not invertible, so $\det(A - I_n) = 0$. Thus 1 is an eigenvalue of A. □

Remark 1. Note that by the argument above, you can also say that if A is an $n \times n$ matrix such that there is a number λ such that

$$\sum_{i=1}^{n} a_{i,j} = \lambda$$

for all j (i.e. the sum of each column's entries is λ), then λ is an eigenvalue of A. Similarly, if

$$\sum_{j=1}^{n} a_{i,j} = \mu$$

for all i (i.e. the sum of each row's entries is μ), then μ is an eigenvector of A.

Denote by $\Sigma : \mathbb{R}^n \to \mathbb{R}$ the linear transformation satisfying $\Sigma(\vec{v}) = \sum_{i=1}^{n} v_i$. Then Σ is a linear transformation attached to the matrix $\underbrace{\begin{bmatrix}1 & 1 & \cdots & 1\end{bmatrix}}_{n \text{ times}}$.

Lemma 1. Let A be an $n \times n$ Markov matrix. Then $\Sigma(A\vec{v}) = \Sigma(\vec{v})$ for all $\vec{v} \in \mathbb{R}^n$.

Proof. Let $\{\vec{e}_1, \ldots, \vec{e}_n\}$ be the standard basis of $\mathbb{R}^n$. Then $\vec{v} = v_1\vec{e}_1 + \cdots + v_n\vec{e}_n$. Let $A = \begin{bmatrix}\vec{a}_1 & \cdots & \vec{a}_n\end{bmatrix}$, so $\Sigma(\vec{a}_i) = 1$ for all i. So $\Sigma(A\vec{v}) = v_1\Sigma(\vec{a}_1) + \cdots + v_n\Sigma(\vec{a}_n) = v_1 + \cdots + \vec{v}_n = \Sigma(\vec{v})$, by linearity of Σ. □

Now suppose $\lambda \neq 1$ is an eigenvalue of A. Then there is an eigenvector $\vec{v}$ such that $A\vec{v} = \lambda v$. Then

$$\Sigma(\vec{v}) = \Sigma(A\vec{v}) = \Sigma(\lambda\vec{v}) = \lambda\Sigma(\vec{v}).$$

Given $\lambda \neq 1$, this equation can only hold if $\Sigma(\vec{v}) = 0$.

Now define the function $\max : \mathbb{R}^n \to \mathbb{R}$ by $\max(\vec{v}) = \max(v_i)$, and similarly define $\min : \mathbb{R}^n \to \mathbb{R}$.

Lemma 2. Let A be a Markov matrix and let $\vec{v} \in \mathbb{R}^n$. Then $\max(A^T\vec{v}) \leq \max(\vec{v})$ and $\min(A^T\vec{v}) \geq \min(\vec{v})$.

Proof. Note that the i^{th} component of $A^T\vec{v}$ is

$$\begin{aligned} a_{1,i}v_1 + a_{2,i}v_2 + \cdots + a_{n,i}v_n &\leq a_{1,i}\max(\vec{v}) + a_{2,i}\max(\vec{v}) + \cdots + a_{n,i}\max(\vec{v}) \\ &= (a_{1,i} + \cdots + a_{n,i})\max(\vec{v}) = \max(\vec{v}). \end{aligned}$$

□

The proof is similar for $\min(A^T\vec{v}) \geq \min(\vec{v})$.

Corollary 1. Let A be an $n \times n$ Markov matrix and let λ be an eigenvalue of A^T. Then $|\lambda| \leq 1$.

Proof. First suppose $\lambda > 1$. Then there is an eigenvector $\vec{v}$ such that $A^T\vec{v} = \lambda\vec{v}$, but then $\max(A^T\vec{v}) = \max(\lambda\vec{v}) = \lambda\max(\vec{v}) > \max(\vec{v})$, which contradicts Lemma 2. Same goes for the case $\lambda < -1$. □

Corollary 2. Let A be an $n \times n$ Markov matrix and let λ be an eigenvalue of A. Then $|\lambda| \leq 1$.

Proof. The eigenvalues of A are the same as the eigenvalues of A^T. □

We can also model dynamical systems that have "memory" by encoding each sequence of states as its own new state. For example, below is a Markov matrix modeling the dynamics of a three-state system with a memory of two.

$$\begin{pmatrix} 8 & 8 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 4 & 7 \\ 1 & 1 & 6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 1 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & 2 & 2 \\ 1 & 1 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 8 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & 4 & 1 \end{pmatrix} /10$$

The steady state vector is

$$\begin{pmatrix} 23669 \\ 4301 \\ 12930 \\ 10555 \\ 8523 \\ 5983 \\ 6676 \\ 12237 \\ 7664 \end{pmatrix},$$

rescaled to sum to 1 is

$$\begin{pmatrix} 23669 \\ 4301 \\ 12930 \\ 10555 \\ 8523 \\ 5983 \\ 6676 \\ 12237 \\ 7664 \end{pmatrix} /92538 = \begin{pmatrix} \frac{23669}{92538} \\ \frac{4301}{92538} \\ \frac{2155}{15423} \\ \frac{10555}{92538} \\ \frac{947}{10282} \\ \frac{5983}{92538} \\ \frac{3338}{46269} \\ \frac{4079}{30846} \\ \frac{3832}{46269} \end{pmatrix} \approx \begin{pmatrix} 0.2557760055 \\ 0.0464782035 \\ 0.1397263827 \\ 0.1140612505 \\ 0.0921027038 \\ 0.0646545203 \\ 0.0721433357 \\ 0.1322375673 \\ 0.0828200307 \end{pmatrix}$$

So we can say that the average probability of the system being in state 1 is

$$\frac{23669 + 4301 + 12930}{92538} = \frac{40900}{92538} \approx 44.19805918\%,$$

for state 2

$$\frac{10555 + 8523 + 5983}{92538} = \frac{25061}{92538} \approx 27.08184746\%$$

and for state 3

$$\frac{6676+12237+7664}{92538}=\frac{26577}{92538}\approx 28.72009337\%.$$

We can also say that the average state 1 has the state vector

$$\begin{pmatrix}23669/40900\\4301/40900\\12930/40900\end{pmatrix}\approx\begin{pmatrix}0.5787041565\\0.1051589242\\0.3161369193\end{pmatrix},$$

and state 2

$$\begin{pmatrix}10555/25061\\8523/25061\\5983/25061\end{pmatrix}\approx\begin{pmatrix}0.4211723395\\0.34009018\\0.2387374805\end{pmatrix},$$

and state 3

$$\begin{pmatrix}6676/26577\\12237/26577\\7664/26577\end{pmatrix}\approx\begin{pmatrix}0.251194642\\0.4604357151\\0.2883696429\end{pmatrix}.$$

This is useful if we want to have an initial state be "state 1" but we have no data for the preceding state. For example,

$$\begin{pmatrix}0.8&0.8&0.1&0&0&0&0&0&0\\0&0&0&0.1&0.1&0.4&0&0&0\\0&0&0&0&0&0&0.4&0.4&0.7\\0.1&0.1&0.6&0&0&0&0&0&0\\0&0&0&0.5&0.1&0.4&0&0&0\\0&0&0&0&0&0&0.3&0.2&0.2\\0.1&0.1&0.3&0&0&0&0&0&0\\0&0&0&0.4&0.8&0.2&0&0&0\\0&0&0&0&0&0&0.3&0.4&0.1\end{pmatrix}\begin{pmatrix}23669/40900\\4301/40900\\12930/40900\\0\\0\\0\\0\\0\\0\end{pmatrix}$$

$$=\begin{pmatrix}\frac{23669}{40900}\\0\\0\\\frac{2111}{8180}\\0\\0\\\frac{1669}{10225}\\0\\0\end{pmatrix}\approx\begin{pmatrix}0.5787041565\\0\\0\\0.2580684597\\0\\0\\0.1632273839\\0\\0\end{pmatrix}$$

and

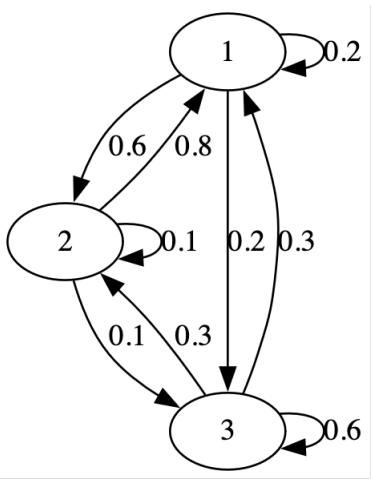
$$\begin{pmatrix} 0.8 & 0.8 & 0.1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.1 & 0.1 & 0.4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.4 & 0.4 & 0.7 \\ 0.1 & 0.1 & 0.6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0.1 & 0.4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.3 & 0.2 & 0.2 \\ 0.1 & 0.1 & 0.3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.4 & 0.8 & 0.2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.3 & 0.4 & 0.1 \end{pmatrix}^2 \begin{pmatrix} 23669/40900 \\ 4301/40900 \\ 12930/40900 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 189352 \\ 10555 \\ 26704 \\ 23669 \\ 52775 \\ 20028 \\ 23669 \\ 42220 \\ 20028 \end{pmatrix} /409000 \approx \begin{pmatrix} 0.4629633252 \\ 0.0258068460 \\ 0.0652909535 \\ 0.0578704156 \\ 0.1290342298 \\ 0.0489682152 \\ 0.0578704156 \\ 0.1032273839 \\ 0.0489682152 \end{pmatrix}$$

and after three steps we get

$$\begin{pmatrix} 1625960 \\ 156556 \\ 403752 \\ 360131 \\ 251232 \\ 195503 \\ 280019 \\ 556932 \\ 259915 \end{pmatrix} /4090000 \approx \begin{pmatrix} 0.3975452323 \\ 0.0382777506 \\ 0.0987168704 \\ 0.0880515892 \\ 0.0614259169 \\ 0.0478002445 \\ 0.0684643032 \\ 0.1361691932 \\ 0.0635488998 \end{pmatrix}.$$

We can also turn a Markov matrix from a discrete dynamical system into a continuous dynamical system (i.e. a system of differential equations). Consider the following Markov graph,



which corresponds to the matrix

$$M = \begin{pmatrix} 0.2 & 0.8 & 0.1 \\ 0.6 & 0.1 & 0.1 \\ 0.2 & 0.1 & 0.8 \end{pmatrix}.$$

The steady state vector of M is

$$\begin{pmatrix} 17 \\ 14 \\ 24 \end{pmatrix} \sim \begin{pmatrix} 17/55 \\ 14/55 \\ 24/55 \end{pmatrix} = \begin{pmatrix} 0.3\overline{09} \\ 0.2\overline{54} \\ 0.4\overline{36} \end{pmatrix}$$

We may use this Markov graph to construct the system of differential equations

$$\begin{aligned} \frac{\mathsf{d}x}{\mathsf{d}t} &= -0.8x + 0.8y + 0.3z \\ \frac{\mathsf{d}y}{\mathsf{d}t} &= 0.6x - 0.9y + 0.3z \\ \frac{\mathsf{d}z}{\mathsf{d}t} &= 0.2x + 0.1y - 0.4z \end{aligned}$$

or

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = (M - I_3) \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Note

$$M - I_3 = \begin{pmatrix} \frac{17}{24} & \frac{-\sqrt{141}-13}{2} & \frac{\sqrt{141}-13}{2} \\ \frac{7}{12} & \frac{\sqrt{141}+11}{2} & \frac{-\sqrt{141}+11}{2} \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{-\sqrt{141}-19}{20} & 0 \\ 0 & 0 & \frac{\sqrt{141}-19}{20} \end{pmatrix} \begin{pmatrix} \frac{17}{24} & \frac{-\sqrt{141}-13}{2} & \frac{\sqrt{141}-13}{2} \\ \frac{7}{12} & \frac{\sqrt{141}+11}{2} & \frac{-\sqrt{141}+11}{2} \\ 1 & 1 & 1 \end{pmatrix}^{-1}$$

and so the solutions are

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = c_1 \begin{pmatrix} \frac{17}{24} \\ \frac{7}{12} \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} \frac{-\sqrt{141}-13}{2} \exp(\frac{-\sqrt{141}-19}{20}t) \\ \frac{\sqrt{141}+11}{2} \exp(\frac{-\sqrt{141}-19}{20}t) \\ \exp(\frac{-\sqrt{141}-19}{20}t) \end{pmatrix} + c_3 \begin{pmatrix} \frac{\sqrt{141}-13}{2} \exp(\frac{\sqrt{141}-19}{20}t) \\ \frac{-\sqrt{141}+11}{2} \exp(\frac{\sqrt{141}-19}{20}t) \\ \exp(\frac{\sqrt{141}-19}{20}t) \end{pmatrix}$$

We can also consider open systems with "fonts" and "sinks." Consider a font of $+0.05$ at vertex 1 and a sink of -0.05 at vertex 3. Then we can build a discrete dynamical system

$$v \mapsto Mv + \underbrace{\begin{pmatrix} 0.05 \\ 0 \\ -0.05 \end{pmatrix}}_{b}.$$

Then the limit of this dynamical system is

$$\lim_{t\to\infty} M^t v + M^{t-1}b + M^{t-2}b + \cdots + Mb + b.$$

Note that for a Markov matrix M, $\lim_{t\to\infty} M^t + \cdots M + I$ does not exist. But I believe for vectors b whose entries add up to 0 (i.e. b is orthogonal to $\vec{\mathbf{1}}$, $\lim_{t\to\infty}(M^t + \cdots + M + I)b$ exists. Note that when b is orthogonal to $\vec{\mathbf{1}}$, then b has no steady-state component of M. This is because $M^t b$ preserves the same sum of entries as b (that is, 0), while also $\lim_{t\to\infty} M^t b$ is a steady state vector whose sum of entries is 0, so $\lim_{t\to\infty} M^t b = \vec{0}$. Then we can write $b = c_1e_1 + c_2e_2 + \cdots + c_ne_n$ where e_i is an eigenvector of M other than a steady state. Denote by λ_i the eigenvalue corresponding to e_i. Then $M^t b = \lambda_1^t c_1 e_1 + \lambda_2^t c_2 e_2 + \cdots + \lambda_n^t c_n e_n$. This creates a convergent geometric series out of $(M^t + \cdots + M + I)b$, since $|\lambda_i| < 1$. Particularly,

$$M^{(\infty)}b := \lim_{t\to\infty}(M^t + \cdots + M + I)b = \frac{c_1}{1-\lambda_1}e_1 + \frac{c_2}{1-\lambda_2}e_2 + \cdots + \frac{c_n}{1-\lambda_n}e_n.$$

Thus,

$$\lim_{t\to\infty} v_t = M^\infty v_0 + M^{(\infty)}b.$$

In our case,

$$b = \underbrace{\frac{11\sqrt{141} - 141}{5640}}_{c_1}\underbrace{\begin{pmatrix} \frac{-\sqrt{141}-13}{2} \\ \frac{\sqrt{141}+11}{2} \\ 1 \end{pmatrix}}_{e_1} + \underbrace{\frac{-11\sqrt{141} - 141}{5640}}_{c_2}\underbrace{\begin{pmatrix} \frac{\sqrt{141}-13}{2} \\ \frac{-\sqrt{141}+11}{2} \\ 1 \end{pmatrix}}_{e_2}$$

and so

$$\frac{c_1}{1-\lambda_1} = \frac{35\sqrt{141} - 423}{6204}$$

and

$$\frac{c_2}{1-\lambda_2} = \frac{-35\sqrt{141} - 423}{6204}$$

and so

$$M^{(\infty)}b = \frac{c_1}{1-\lambda_1}e_1 + \frac{c_2}{1-\lambda_2}e_2 = \begin{pmatrix} \frac{-8*\sqrt{141}+141}{3102} \\ \frac{-19*\sqrt{141}+141}{6204} \\ \frac{35*\sqrt{141}-423}{6204} \end{pmatrix} + \begin{pmatrix} \frac{8*\sqrt{141}+141}{3102} \\ \frac{19*\sqrt{141}+141}{6204} \\ \frac{-35*\sqrt{141}-423}{6204} \end{pmatrix} = \begin{pmatrix} \frac{1}{11} \\ \frac{1}{22} \\ \frac{-3}{22} \end{pmatrix}$$

So now, we can find the limit if we start with an initial state $v_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Since $M^{\infty}v = \begin{pmatrix} 17/55 \\ 14/55 \\ 24/55 \end{pmatrix}$ from earlier, and $M^{(\infty)}b = \begin{pmatrix} 1/11 \\ 1/22 \\ -3/22 \end{pmatrix}$, we get

$$\lim_{t\to\infty} v_t = \begin{pmatrix} 17/55 \\ 14/55 \\ 24/55 \end{pmatrix} + \begin{pmatrix} 1/11 \\ 1/22 \\ -3/22 \end{pmatrix} = \begin{pmatrix} 22/55 \\ 33/110 \\ 33/110 \end{pmatrix} = \begin{pmatrix} 2/5 \\ 3/10 \\ 3/10 \end{pmatrix} = \begin{pmatrix} 0.4 \\ 0.3 \\ 0.3 \end{pmatrix}.$$

Making a slight modification, we can also see

$$M^{(\infty)} \begin{pmatrix} 0 \\ 0.05 \\ -0.05 \end{pmatrix} = \begin{pmatrix} 7/110 \\ 9/110 \\ -8/55 \end{pmatrix}$$

and so

$$\lim_{t\to\infty} v_t = \begin{pmatrix} 17/55 \\ 14/55 \\ 24/55 \end{pmatrix} + \begin{pmatrix} 7/110 \\ 9/110 \\ -8/55 \end{pmatrix} = \begin{pmatrix} 41/110 \\ 37/110 \\ 16/55 \end{pmatrix} = \begin{pmatrix} 0.3\overline{72} \\ 0.3\overline{36} \\ 0.2\overline{90} \end{pmatrix}.$$

A simpler way to compute this limit is that $\lim_{t\to\infty} v_t$ is in the solution set $(I - M) \setminus b$ where the sum of v_∞'s entries equals that of v_0's.

Let us see an example in differential equations:

$$X' = \left(\begin{pmatrix} 0.6 & 0.5 & 0.2 \\ 0.2 & 0.2 & 0.6 \\ 0.2 & 0.3 & 0.2 \end{pmatrix} - I_3 \right) X + \begin{pmatrix} -0.05 \\ 0 \\ 0.05 \end{pmatrix},$$

where

$$A = M - I_3 = \begin{pmatrix} 23/11 & 1 & -3 \\ 14/11 & -2 & 2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1.2 & 0 \\ 0 & 0 & -0.8 \end{pmatrix} P^{-1}.$$

Then by defining $Y = P^{-1}X$, we have

$$Y' = DY + P^{-1}b = DY + \begin{pmatrix} 0 \\ 1/40 \\ 1/40 \end{pmatrix}.$$

And so we get a system of differential equations:

$$\begin{aligned} y_1' &= 0, \\ y_2' &= -1.2y_2 + 1/40, \\ y_3' &= -0.8y_3 + 1/40. \end{aligned}$$

The solutions are

$$Y = \begin{pmatrix} C_1 \\ -\frac{1}{24}\exp(-0.6t)\sinh(-0.6t) + C_2\exp(-1.2t) \\ -\frac{1}{16}\exp(-0.4t)\sinh(-0.4t) + C_2\exp(-0.8t) \end{pmatrix}.$$

In general, the solution to $y' = ay + b$ is $y = \frac{2b}{a}\exp(at/2)\sinh(at/2) + y(0)\exp(at)$. Finally,

$$X = PY$$

is our solution set. Note that $\exp(at)\sinh(at) = \frac{\exp(2a)-1}{2}$, so

$$\lim_{t\to\infty} \exp(at)\sinh(at) = -\frac{1}{2}$$

for $a < 0$, and so the steady state vectors can be found as

$$\lim_{t\to\infty} PY(t) = P\begin{pmatrix} C_1 \\ 1/48 \\ 1/32 \end{pmatrix} = \begin{pmatrix} 23C_1/11 - 7/96 \\ 14C_1/11 + 1/48 \\ C_1 + 5/96 \end{pmatrix}.$$

Here is a demo. Choosing $C_1 = 11/48$ gives us a steady state vector of 1-norm 1. Then we have

$$\begin{pmatrix} 13/32 \\ 5/16 \\ 9/32 \end{pmatrix} = \begin{pmatrix} 0.40625 \\ 0.3125 \\ 0.28125 \end{pmatrix},$$

as opposed to the corresponding steady-state for the linear system of equations ($b = 0$), which is

$$\begin{pmatrix} 23/48 \\ 7/24 \\ 11/48 \end{pmatrix} = \begin{pmatrix} 0.4791\overline{6} \\ 0.291\overline{6} \\ 0.2291\overline{6} \end{pmatrix}.$$

What if $b = \begin{pmatrix} -0.05 \\ 0 \\ 0.05 + 0.02\sin(t) \end{pmatrix}$? Note that $P^{-1}b = \begin{pmatrix} \frac{11*\sin(t)}{2400} \\ \frac{\sin(t)+4}{160} \\ \frac{11*\sin(t)+30}{1200} \end{pmatrix}$. Then our system of differential equations becomes

$$\begin{aligned} y_1' &= 11\sin(t)/2400, \\ y_2' &= -1.2y_2 + \sin(t)/160 + 1/40, \\ y_3' &= -0.8y_3 + 11\sin(t)/1200 + 1/40. \end{aligned}$$

The solution to

$$y' = ay + b\sin(t) + c$$

is

$$y(t) = -(ab\cos(t))/(a^3 + a) - c/(a^3 + a) - (a^2 b\sin(t))/(a^3 + a) - (a^2 c)/(a^3 + a) + C\exp(at)$$

when $a \neq 0$, or

$$y(t) = \frac{-ab\cos(t) - c - a^2c - a^2b\sin(t)}{a^3 + a} + C\exp(at).$$

So

$$\begin{aligned}
y_1 &= -11\cos(t)/2400 + C_1 \\
y_2 &= -\frac{5}{1952}\cos(t) + \frac{1}{48} + \frac{3}{976}\sin(t) + C_2\exp(-6t/5) \\
y_3 &= -\frac{11}{1968}\cos(t) + \frac{1}{32} + \frac{11}{2460}\sin(t) + C_3\exp(-4t/5)
\end{aligned}$$

Here is a demo of the solutions.