Let k be an algebraically closed field and let $F \in k[x_0, x_1, \dots, x_n]$ be a homogeneous polynomial satisfying $F(1, 0, \dots, 0) = 0$. Let $V(F)$ be the corresponding variety. Let $\deg F = d$.

We know that $P = [1 : 0 : \cdots : 0] \in V(F)$ and so F has the form

$$F = x_0^{d-1} F_1(x_1, \dots, x_n) + x_0^{d-2} F_2(x_1, \dots, x_n) + \cdots + x_0 F_{d-1}(x_1, \dots, x_n) + F_d(x_1, \dots, x_n)$$

with each F_i homogeneous.

We wish to show that $Y = \bigcap_{i=1}^{d} V(F_d) \subseteq \mathbb{P}_k^{n-1}$ parametrizes the set of lines ℓ satisfying $P \in \ell \subseteq V(F)$.

First let $[a_1 : \cdots : a_n] \in Y$. Then we claim that the line connecting P and $[1 : a_1 : \cdots : a_n]$ is contained in $V(F)$. Note that the line ℓ connecting P and $[1 : a_1 : \cdots : a_n]$ can be parametrized by $[u : ta_1 : \cdots : ta_n]$ for $[u : t] \in \mathbb{P}_k^1$. Then note that

$$\begin{aligned} F(u, ta_1, \dots, ta_n) &= u^{d-1} F_1(ta_1, \dots, ta_n) + \cdots + u F_{d-1}(ta_1, \dots, ta_n) + F_d(ta_1, \dots, ta_n) \\ &= u^{d-1} t F_1(a_1, \dots, a_n) + \cdots + u t^{d-1} F_{d-1}(a_1, \dots, a_n) + t^d F_d(a_1, \dots, a_n) = 0 \end{aligned}$$

since $F_i(a_1, \dots, a_n) = 0$ for all $1 \le i \le d$. Thus every point on ℓ is in $V(F)$ and so $\ell \subseteq V(F)$.

Also note that this correspondence is unique because different points in $\mathbb{P}_k^{n-1}$ will correspond to different lines in $\mathbb{P}_k^n$ through P.

Now let us show the reverse direction. Let ℓ be a line in $\mathbb{P}_k^n$ such that $P \in \ell \subseteq V(F)$. We wish to show that ℓ corresponds to some point in $Y \subseteq \mathbb{P}^{n-1}$.

Let $P \neq [1 : a_1 : \cdots : a_n] \in \ell$. We wish to show that $F_i(a_1, \dots, a_n) = 0$ for all $1 \le i \le d$. Since $P, [1 : a_1 : \cdots : a_n] \in \ell$, we know once again that ℓ can be parametrized by $[u : t] \in \mathbb{P}_k^1$ as $[u : ta_1 : \cdots : ta_n]$. Since $\ell \subseteq V(F)$, we know that $F(u, ta_1, \dots, ta_n) = 0$ for all $[u : t] \in \mathbb{P}_k^1$. Thus

$$0 = F(u, ta_1, \dots, ta_n) = u^{d-1} t F_1(a_1, \dots, a_n) + \cdots + u t^{d-1} F_{d-1}(a_1, \dots, a_n) + t^d F_d(a_1, \dots, a_n)$$

for all $[u : t] \in \mathbb{P}_k^1$. This is clear when $t = 0$. When $t \neq 0$ we have

$$0 = u^{d-1} F_1(a_1, \dots, a_n) + \cdots + u F_{d-1}(a_1, \dots, a_n) + F_d(a_1, \dots, a_n) \in k[u].$$

This polynomial can only be 0 when all of the coefficients are 0. Thus we know that $F_i(a_1, \dots, a_n) = 0$ for all $1 \le i \le d$. Therefore ℓ corresponds to $[a_1 : \cdots : a_n] \in Y$.

Thus we have our conclusion that Y parametrizes the set of lines ℓ satisfying $P \in \ell \subseteq V(F)$. Thus note that when Y is empty, we know that no line going through P is contained in $V(F)$.