Part 1 is from the paper "Lines in $\mathbb{P}^3$." Let $\operatorname{char} k = 0$. Points in $\mathbb{P}^3$ correspond to (projective equivalence classes) of nonzero vectors in k^4. That is, the point in $\mathbb{P}^3$ with homogeneous coordinates $[X : Y : Z : W]$ is the line $[\mathbf{v}]$ spanned by the nonzero vector

$$\mathbf{v} := \begin{bmatrix} X \\ Y \\ Z \\ W \end{bmatrix} \in k^4.$$

Similarly, planes in $\mathbb{P}^3$ correspond to (projective equivalence classes) of covectors

$$\varphi := \begin{bmatrix} a & b & c & d \end{bmatrix} \in (k^4)^*,$$

where $[\varphi] = [\![a : b : c : d]\!]$ is the hyperplane defined in homogeneous coordinates by $\varphi(\mathbf{v}) = 0$, that is,

$$aX + bY + cZ + dW = 0. \tag{1}$$

That is, the point $[X : Y : Z : W]$ lies on the plane $[\![a : b : c : d]\!]$ if and only if (1) is satisfied.

Thus points and planes in $\mathbb{P}^3$ are defined in homogeneous coordinates by vectors in the vector space $V := k^4$ and covectors in its dual vector space $V^* = (k^4)^*$. Moreover, the orthogonal complement $\mathbf{v}^\perp$ of the line $k\mathbf{v} \subseteq k^4$ is the hyperplane in k^4 defined by the covector $\mathbf{v}^\dagger$, which is the conjugate transpose of $\mathbf{v}$.

How can you describe *lines* in $\mathbb{P}^3$ in a similar way by homogeneous coordinates?

Exterior Outer Products

Recall that $\mathfrak{so}(n)$ denotes the set of $n \times n$ skew-symmetric matrices. That is, $X \in \mathrm{Mat}_n$ such that $X + X^\dagger = 0$. The *exterior outer product* is the alternating bilinear map:

$$\begin{aligned} k^n \times k^n &\to \mathfrak{so}(n) \\ (u, v) &\mapsto u \wedge v := v^\dagger u - u^\dagger v. \end{aligned}$$

We shall verify the following:

(1) $(u \wedge v)(w) = (u \cdot w)v - (v \cdot w)u$.

(2) If $n = 3$, then $(u \wedge v)(w) = (u \times v) \times w$.

(3) The vectors u and v are linearly dependent if and only if $u \wedge v = 0$.

(4) If u and v are linearly independent, then the projective equivalence class $[u \wedge v] \in \mathbb{P}(\mathfrak{so}(n))$ depends only on the plane $k\langle u, v\rangle$.

(5) The orthogonal complement of the plane $k\langle u, v\rangle \subseteq V$ lies in the kernel $\ker(u \wedge v)$. In other words,
$$k\langle u, v\rangle^\perp \subseteq \ker(u \wedge v).$$

Proof.

(1) First let $u = (u_1, u_2, \ldots, u_n)$ and let $v = (v_1, v_2, \ldots, v_n)$. Then

$$u \wedge v = \begin{pmatrix} u_1v_1 & u_2v_1 & \cdots & u_nv_1 \\ u_1v_2 & u_2v_2 & \cdots & u_nv_2 \\ \vdots & \vdots & \ddots & \vdots \\ u_1v_n & u_2v_n & \cdots & u_nv_n \end{pmatrix} - \begin{pmatrix} u_1v_1 & u_1v_2 & \cdots & u_1v_n \\ u_2v_1 & u_2v_2 & \cdots & u_2v_n \\ \vdots & \vdots & \ddots & \vdots \\ u_nv_1 & u_nv_2 & \cdots & u_nv_n \end{pmatrix}$$
$$= \begin{pmatrix} 0 & u_2v_1 - u_1v_2 & \cdots & u_nv_1 - u_1v_n \\ u_1v_2 - u_2v_1 & 0 & \cdots & u_nv_2 - u_2v_n \\ \vdots & \vdots & \ddots & \vdots \\ u_1v_n - u_nv_1 & u_2v_n - u_nv_2 & \cdots & 0 \end{pmatrix}$$

Then let $w = (w_1, w_2, \ldots, w_n)$. Then

$$(u \wedge v)(w) = \left(\sum_{i=1}^{n} (u_iv_1 - u_1v_i)w_i, \ldots, \sum_{i=1}^{n} (u_iv_n - u_nv_i)w_i \right)$$
$$= \left(\sum_{i=1}^{n} u_iw_iv_1, \ldots, \sum_{i=1}^{n} u_iw_iv_n \right) - \left(\sum_{i=1}^{n} v_iw_iu_1, \ldots, \sum_{i=1}^{n} v_iw_iu_n \right)$$
$$= \left(\sum_{i=1}^{n} u_iw_i \right) v - \left(\sum_{i=1}^{n} v_iw_i \right) u = (u \cdot w)v - (v \cdot w)u.$$

(2) Now let $n = 3$. Then

$$(u \wedge v)(w) = (u \cdot w)v - (v \cdot w)u$$
$$= (u_1w_1 + u_2w_2 + u_3w_3)v - (v_1w_1 + v_2w_2 + v_3w_3)u = (a, b, c)$$

where

$$a = u_2v_1w_2 + u_3v_1w_3 - u_1v_2w_2 - u_1v_3w_3$$
$$b = u_1v_2w_1 + u_3v_2w_3 - u_2v_1w_1 - u_2v_3w_3$$
$$c = u_1v_3w_1 + u_2v_3w_2 - u_3v_1w_1 - u_3v_2w_2.$$

Now note that $u \times v = (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1) =: (x, y, z)$. Then $(x, y, z) \times w = (yw_3 - zw_2, zw_1 - xw_3, xw_2 - yw_1)$. Now note that

$$a = yw_3 - zw_2$$
$$b = zw_1 - xw_3$$
$$c = xw_2 - yw_1$$

and so $(u \wedge v)(w) = (u \times v) \times w$.

(3) ($\Rightarrow$) Suppose u and v are linearly dependent. Thus there are $a, b \in k^\times$ such that $au + bv = 0$. Thus $v = cu$ where $c = -a/b$. Then $u \wedge v = (cu)^\dagger u - u^\dagger(cu) = c(u^\dagger u - u^\dagger u) = 0$.

($\Leftarrow$) Now suppose $u \wedge v = 0$. Then we know from (2) that $(u \cdot w)v = (v \cdot w)u$ for all $w \in k^n$, including $w \in k\langle u, v\rangle$ where $u \cdot w \neq 0$ or $v \cdot w \neq 0$. Thus u and v are linearly dependent.

(4) Let u and v be linearly independent. Then $k\langle u, v\rangle$ is a plane in k^n. Let $a, b, c, d \in k$ be such that $au + bv, cu + dv \in k\langle u, v\rangle$ are linearly independent vectors. We will show that $u \wedge v \sim (au + bv) \wedge (cu + dv) \in \mathbb{P}(\mathfrak{so}(n))$. Note that $\dim \mathbb{P}(\mathfrak{so}(n)) = \frac{n^2-n}{2} - 1$.

Let $w \in k^n$. Then

$$\begin{aligned} &((au + bv) \cdot w)(cu + dv) - ((cu + dv) \cdot w)(au + bv) \\ &= (a(u \cdot w) + b(v \cdot w))(cu + dv) - (c(u \cdot w) + d(v \cdot w))(au + bv) \\ &= (ad - bc)(u \cdot w)v + (bc - ad)(v \cdot w)u = (ad - bc)((u \cdot w)v - (v \cdot w)u) \\ &= (ad - bc)(u \wedge v)(w). \end{aligned}$$

Therefore $(au+bv)\wedge(cu+dv) = (ad-bc)(u\wedge v)$ and so $[u\wedge v] = [(au+bv)\wedge(cu+dv)]$ in $\mathbb{P}(\mathfrak{so}(n))$.

Now let $s, t \in k^n$ be linearly independent vectors and suppose there is a $\lambda \in k^\times$ such that $s \wedge t = \lambda(u \wedge v)$. Then for all $w \in k^n$, we have

$$(s \cdot w)t - (t \cdot w)s = \lambda((u \cdot w)v - (v \cdot w)u) = (\lambda u \cdot w)v - (\lambda v \cdot w)u \in k\langle u, v\rangle.$$

Since this is true for all $w \in k^n$, we have $s, t \in k\langle u, v\rangle$. For example, since s and t are linearly independent, we may select a w such that $s \cdot w = 0$ and $t \cdot w \neq 0$ and vice versa. Therefore $k\langle s, t\rangle = k\langle u, v\rangle$. Thus the equivalence class of $[u \wedge v]$ in $\mathbb{P}(\mathfrak{so}(n))$ depends solely on the plane $k\langle u, v\rangle$.

(5) Let $z \in k\langle u, v\rangle^\perp$. Thus $z \cdot u = z \cdot v = 0$. Thus $(u \wedge v)(z) = (u \cdot z)v - (v \cdot z)u = 0$ and so $z \in \ker(u \wedge v)$. Thus $k\langle u, v\rangle^\perp \subseteq \ker(u \wedge v)$.

$\square$

Because of (4), every 2-dimensional linear subspace L (plane through the origin) of k^n determines an element of the projective space $\mathbb{P}(\mathfrak{so}(n))$; the corresponding homogeneous coordinates are called the *Plücker coordinates* of the plane, or the corresponding projective line $\mathbb{P}(L) \subseteq \mathbb{P}(k^n)$.

Plücker coordinates in $\mathbb{P}^3$

Let $V = k^4$ and $\Lambda = \mathfrak{so}(4)$ be the 6-dimensional vector space of 4×4 skew-symmetric matrices. Then lines in $\mathbb{P}^3 = \mathbb{P}(V)$ correspond to 2-dimensional linear subspaces of V, which in turn correspond to projective equivalence classes of certain nonzero elements $u \wedge v \in \Lambda$. Which elements of Λ correspond to lines in $\mathbb{P}^3$?

Since $\dim V = 4$, the plane $k\langle u, v\rangle \neq V$, so there is a nonzero vector n normal to this plane. By the above, n lies in the kernel of the skew-symmetric matrix $u \wedge v$. Thus lines in $\mathbb{P}^3$ determine nonzero singular (non-injective) matrices in Λ. Note that

$$\det\begin{pmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{pmatrix} = c^2d^2 + b^2e^2 - 2bcde + a^2f^2 + 2acdf - 2abef = (af - be + cd)^2.$$

When n is even, skew-symmetric matrices in $\mathfrak{so}(n)$ have the following property. In general the determinant of an $n \times n$ is a degree n polynomial in its entries. When n is even, there is a degree $n/2$ polynomial $\mathscr{P}$ on $\mathfrak{so}(n)$ (called the *Pfaffian*) such that if $M \in \mathfrak{so}(n)$, then

$$\det M = \mathscr{P}(M)^2.$$

That is, in even dimensions, the determinant of a skew-symmetric matrix is a perfect square. For example, then $n = 2$, the general skew-symmetric matrix is

$$M = \begin{pmatrix} 0 & y \\ -y & 0 \end{pmatrix},$$

which has determinant y^2. Thus $\mathscr{P}(M) = y$.

When $n = 4$, the Pfaffian is a quadratic polynomial. We've seen above that

$$\det\begin{pmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{pmatrix} = (af - be + cd)^2$$

so the Pfaffian is

$$\mathscr{P}(M) = af - be + cd.$$

The vector space Λ has dimension 6, with coordinates

$$a, b, c, d, e, f.$$

Thus projective equivalence classes of nonzero 4×4 skew-symmetric matrices is the projective space

$$\mathbb{P}(\Lambda) \cong \mathbb{P}^5$$

with homogeneous coordinates

$$[a : b : c : d : e : f].$$

The nonzero singular matrices (namely, those of rank two), are those for which $\mathscr{P}(M) = 0$, which is just the homogeneous quadratic polynomial condition:

$$af - be + cd = 0.$$

This defines a *quadratic hypersurface* $\mathcal{Q}$ in $\mathbb{P}^5$. Since it is defined by one equation in a 5-dimensional space, this quadratic has dimension 4.

Intuitively, we would expect that the space of lines in $\mathbb{P}^3$ has dimension 4. A generic line $\ell \subseteq \mathbb{P}^3$ is not ideal and does not pass through $(0:0:0:1)$. In that case, there is a point

$$p(\ell) \in \mathbb{A}^3 \setminus \{0\}$$

closest to the origin $0 \in k^3$. These points form a 3-dimensional space $k^3 \setminus \{0\}$.

Any point $p \in k^3 \setminus \{0\}$ is the closest point $p(\ell)$ for some ℓ. Namely, look at the plane $W(p)$ containing p and is normal to the vector from 0 to p. Any line ℓ on $W(p)$ passing through p satisfies $p(\ell) = p$. The set of all lines $\ell \subseteq W(p)$ passing through p forms a $\mathbb{P}^1$, which is one-dimensional. Thus lines in $\mathbb{P}^3$ are parametrized by a $3 + 1 = 4$ dimensional space.

This space is the quadric $\mathcal{Q}$ defined above.

Just as quadric surfaces in $\mathbb{P}^3$ can be parametrized as tori $S^1 \times S^1$, the 4-dimensional quadric hypersurface in $\mathbb{P}^5$ can be parametrized by $S^2 \times S^2$. Namely, make the elementary linear substitution

$$\begin{aligned} X &= (c+d)/2, & A &= (c-d)/2, \\ Y &= (b+e)/2, & B &= (b-e)/2, \\ Z &= (a+f)/2, & C &= (a-f)/2 \end{aligned}$$

so that

$$\begin{aligned} \mathscr{P}(M) &= cd - be + af \\ &= X^2 - A^2 + Y^2 - B^2 + Z^2 - C^2. \end{aligned}$$

Thus $\mathcal{Q}$ is the quadric in $\mathbb{P}^5$ consisting of points with homogeneous coordinates $[X : Y : Z : A : B : C]$ satisfying

$$X^2 + Y^2 + Z^2 = A^2 + B^2 + C^2.$$

By projective rescaling we may assume that $X^2+Y^2+Z^2 = A^2+B^2+C^2 = 1$. Each of these equations describes an S^2. Since the coordinates (A, B, C) and (X, Y, Z) are independent of one another, the quadricn $\mathcal{Q}$ looks like $S^2 \times S^2$.

Orthogonal Complement and Involution

Since $\mathscr{P}$ is a homogeneous quadratic function on the vector space Λ, it arises from a symmetric bilinear form $\mathcal{P}$ on Λ by the usual correspondences:

$$\begin{aligned} \mathscr{P}(X) &= \mathcal{P}(X, X), \\ \mathcal{P}(X, Y) &:= \frac{1}{2}(\mathscr{P}(X+Y) - \mathscr{P}(X) - \mathscr{P}(Y)). \end{aligned}$$

Explicitly,

$$\mathcal{P}(M, N) = \frac{1}{2}(cd' + c'd - be' - b'e + af' + a'f).$$

The usual inner product (dot product) on $\mathfrak{so}(4)$ is given by

$$M \cdot N = -\frac{1}{2}\operatorname{tr}(MN)$$
$$aa' + bb' + cc' + dd' + ee' + ff'.$$

We can construct a linear isomorphism $\mathcal{I} : \Lambda \to \Lambda$ defined by

$$\mathcal{I}(M) = \begin{pmatrix} 0 & f & -e & d \\ -f & 0 & c & -b \\ e & -c & 0 & a \\ -d & b & -a & 0 \end{pmatrix},$$

that is,

$$\mathcal{I}(a, b, c, d, e, f) = (f, -e, d, c, -b, a).$$

So $\mathcal{I} \circ \mathcal{I} = \mathrm{id}_\Lambda$ and $\mathscr{P}(\mathcal{I}(M)) = \mathscr{P}(M)$ and $M \cdot \mathcal{I}(M) = 2af - 2be + 2cd = \mathscr{P}(M) + \mathscr{P}(\mathcal{I}(M))$.

Geometrically, if $M \in \mathcal{Q}$ corresponds to a 2-dimensional linear subspace $L \subsetneq V$, then $\mathcal{I}(M)$ corresponds to the orthogonal complement $L^\perp \subsetneq V$. You can also think of the involution $\mathcal{I}$ as mapping the points on the S^2's:

$$\mathcal{I}(X : Y : Z : A : B : C) = (X : -Y : Z : -A : B : -C).$$

If $p \in \mathbb{P}^3$ is a point corresponding to a 1-dimensional linear subspace $L \subsetneq V$, then its dual plane $p^* \subsetneq \mathbb{P}^3$ corresponds to the orthogonal complement $L^\perp$. The homogeneous coordinates of p^* form the *transpose* of the vector formed by the homogeneous coordinates of p. Then $\mathcal{I}$ maps lines through p to the lines contained in the plane p^*. Note that $(U \cap V)^\perp = U^\perp + V^\perp$ and $(U + V)^\perp = U^\perp \cap V^\perp$.

Here is an example. Take p to be the point $(0 : 0 : 0 : 1)$. Then p^* is the plane $w = 0$. The line through 0 in the direction $(a, b, c, 1)$ has Plücker coordinates

$$M = \begin{pmatrix} 0 & 0 & 0 & a \\ 0 & 0 & 0 & b \\ 0 & 0 & 0 & c \\ -a & -b & -c & 0 \end{pmatrix}.$$

Its dual is a line on the plane $w = 0$, which in the plane $w = 0$ has homogeneous coordinates $[\![a : b : c]\!]$ (that is, the line defined in homogeneous coordinates $aX + bY + cZ = 0$). In $\mathbb{P}^3$ this line has Plücker coordinates

$$\mathcal{I}(M) = \begin{pmatrix} 0 & c & -b & 0 \\ -c & 0 & a & 0 \\ b & -a & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

In general, the Plücker coordinates of the line that connects $u, v \in \mathbb{P}^3$ are

$$u \wedge v = \begin{pmatrix} 0 & u_2v_1 - u_1v_2 & u_3v_1 - u_1v_3 & u_4v_1 - u_1v_4 \\ u_1v_2 - u_2v_1 & 0 & u_3v_2 - u_2v_3 & u_4v_2 - u_2v_4 \\ u_1v_3 - u_3v_1 & u_2v_3 - u_3v_2 & 0 & u_4v_3 - u_3v_4 \\ u_1v_4 - u_4v_1 & u_2v_4 - u_4v_2 & u_3v_4 - u_4v_3 & 0 \end{pmatrix}$$

and the dual of this line is

$$\mathcal{I}(u \wedge v) = \begin{pmatrix} 0 & u_4v_3 - u_3v_4 & u_2v_4 - u_4v_2 & u_3v_2 - u_2v_3 \\ u_3v_4 - u_4v_3 & 0 & u_4v_1 - u_1v_4 & u_1v_3 - u_3v_1 \\ u_4v_2 - u_2v_4 & u_1v_4 - u_4v_1 & 0 & u_2v_1 - u_1v_2 \\ u_2v_3 - u_3v_2 & u_3v_1 - u_1v_3 & u_1v_2 - u_2v_1 & 0 \end{pmatrix}.$$

Idea: The point $w \in \mathbb{P}^3$ is on the dual of the line $u \wedge v$ if $(u \wedge v)(w) = 0$. Recall $k\langle u, v\rangle^\perp \subseteq \ker(u \wedge v)$. Since $n = 4$ and $u \wedge v$ has rank 2, we have equality. So $k\langle u, v\rangle^\perp$ corresponds to $\mathcal{I}(u \wedge v)$.

Idea: $(u \wedge v)\mathcal{I}(u \wedge v) = \mathcal{I}(u \wedge v)(u \wedge v) = 0$. This can be seen by multiplying the matrices and the fact that $\mathscr{P}(u \wedge v) = 0$.

Then we have the following maps:

$$k^4 \xrightarrow{\mathcal{I}(u \wedge v)} k^4 \xrightarrow{u \wedge v} k^4$$

where im $(\mathcal{I}(u \wedge v)) \subseteq \ker(u \wedge v) = k\langle u, v\rangle^\perp$. We have equality because $\mathcal{I}(u \wedge v)$ is rank 2. We can write $k\langle u, v\rangle^\perp = k\langle s, t\rangle$ with $s \cdot u = t \cdot u = s \cdot v = t \cdot v = 0$. Thus im $(\mathcal{I}(u \wedge v)) = k\langle s, t\rangle =$ im $(s \wedge t)$. Thus $\mathcal{I}(u \wedge v) \sim s \wedge t$ in $\mathbb{P}(\mathfrak{so}(4))$ (tenuous, my guess is $\mathcal{I}(u \wedge v)$ must be the exterior product of *something* because its Pfaffian is zero?), so $[\mathcal{I}(u \wedge v)] = [s \wedge t]$.

The idea works with the example of $u = (0, 0, 0, 1)$ and $v = (a, b, c, 1)$ as above. The vectors $(b, -a, 0, 0), (c, 0, -a, 0) \in \ker(u \wedge v)$ and

$$(b, -a, 0, 0) \wedge (c, 0, -a, 0) = \begin{pmatrix} 0 & -ac & ba & 0 \\ ac & 0 & -a^2 & 0 \\ -ba & a^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & c & -b & 0 \\ -c & 0 & a & 0 \\ b & -a & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \mathcal{I}(u \wedge v).$$

Idea: If w is on the line connecting u and v, then so is $(u \wedge v)(w)$. This is because $(u \wedge v)(w) = (u \cdot w)v - (v \cdot w)u \in k\langle u, v\rangle \subseteq k^4$, which corresponds to the line that connects u and v in $\mathbb{P}^3$.

Relation to Orth

The alternating trilinear function Orth is a kind of four-dimensional cross product. It can be defined in terms of the involution $\mathcal{I}$ and exterior product $\wedge$:

$$\mathsf{Orth}(u, v, w) = \mathcal{I}(u \wedge v)(w).$$

Three points $[u], [v], [w] \in \mathbb{P}^3$ (where $u, v, w \in V$ are nonzero vectors) are colinear if and only if $\mathsf{Orth}(u, v, w) = 0$. Otherwise they space a plane in $\mathbb{P}^3$ represented by $[\mathsf{Orth}(u, v, w)^\dagger]$.

Dually, suppose $\varphi, \psi, \xi \in V^*$ are nonzero covectors. The corresponding planes $[\varphi], [\psi], [\xi] \subseteq \mathbb{P}^3$ meet in a single point if and only if φ, ψ, ξ are linearly in dependent, in which case

$$[\varphi] \cap [\psi] \cap [\xi] = [\mathsf{Orth}(\varphi^\dagger, \psi^\dagger, \xi^\dagger)]$$

.

Note that the line that connects $(a : b : c)$ and $(d : e : f)$ in $\mathbb{P}^2$ is

$$(bf - ce)x + (cd - af)y + (ae - bd)z.$$

This is the cross product $(a, b, c) \times (d, e, f)$ in k^3.

Exterior Algebras

4.1: Lines and 2-vectors

Definition 12. An *alternating bilinear form* an a k-vector space V is a map $B : V \times V \to k$ such that

- $B(v, w) = -B(w, v)$;
- $B(\lambda_1 v_1 + \lambda_2 v_2, w) = \lambda_1 B(v_1, w) + \lambda_2 B(v_2, w)$.

This is the skew-symmetric version of the symmetric bilinear forms we used to define quadrics. Given a basis $\{v_1, \ldots, v_n\}$, B is uniquely determined by the skew symmetric matrix $B(v_i, v_j)$. We can add alternating forms and multiply by scalars so they form a vector space, isomorphic to the space of the skew-symmetric $n \times n$ matrices. This has dimension $n(n-1)/2$ spanned by the basis elements E^{ab} for $a < b$ where $E^{ab}_{ij} = 0$ if $\{a, b\} \neq \{i, j\}$ and $E^{ab}_{ab} = -E^{ab}_{ba} = 1$.

Definition 13. The *second exterior power* $\Lambda^2 V$ of a finite-dimensional vector space is the dual space of the vector space of alternating bilinear forms on V. Elements of $\Lambda^2 V$ are called 2-*vectors.*

Definition 14. Given $u, v \in V$, the *exterior product* $u \wedge v \in \Lambda^2 V$ is the linear map to k which, on an alternating bilinear form B, takes the value

$$(u \wedge v)(B) = B(u, v).$$

This conflicts a little bit with the earlier paper. The earlier paper defines $u \wedge v$ as a function from V to V, while this paper defines $u \wedge v$ as a function from the space of all alternating bilinear maps to k. Indeed, given a fixed w, the map $B(u, v) = (u \cdot w)v - (v \cdot w)u$ is an alternating bilinear map. Does the other correspondence hold or this new definition a generalization? Perhaps the other correspondence does not hold, since the space of all alternating bilinear maps is an $n(n-1)/2$-dimensional vector space whereas V is an n-dimensional vector space.

I do not think the correspondence moves the other way because we have

$$[B] \xleftrightarrow{\text{duality}} [u \wedge v] \xleftrightarrow{(4)} k\langle u, v\rangle \longleftrightarrow k\langle w\rangle = k\langle u, v\rangle^{\perp}$$

only when $n = 3$, so $n(n-1)/2 = n$.

From this definition follows some basic properties:

$$(u \wedge v)(B) = B(u, v) = -B(v, u) = -(v \wedge u)(B)$$

so that

$$v \wedge u = -u \wedge v$$

and in particular $u \wedge u = 0$. Also

$$(\lambda_1 u_1 + \lambda_2 u_2) \wedge v = \lambda_1 (u_1 \wedge v) + \lambda_2 (u_2 \wedge v).$$

If $\{v_1, \ldots, v_n\}$ is a basis for V, then $\{v_i \wedge v_j\}_{1 \leq i < j \leq n}$ is a basis for $\Lambda^2 V$.

This last property holds because $v_i \wedge v_j(E^{ab}) = E^{ab}_{ij}$ and in fact shows that $\{v_i \wedge v_j\}$ is the dual basis to the basis $\{E^{ab}\}$.

Another important property is $u \wedge v = 0$ if and only if $v = \lambda u$ for some scalar λ. This is proven above.

It is the elements of $\Lambda^2 V$ of the form $u \wedge v$ which will concern us, for suppose $U \subseteq V$ is a 2-dimensional vector subspace and $\{u, v\}$ is a basis of U. Then any other basis is of the form $\{au + bv, cu + dv\}$ so

$$(au + bv) \wedge (cu + dv) = (ad - bc)(u \wedge v)$$

and since the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible $ad - bc \neq 0$. It follows that the 1-dimensional subspace of $\Lambda^2 V$ spanned by $u \wedge v$ for a basis of U is well-defined by U itself and is independent of choice of basis. To each line in $\mathbb{P}(V)$ we can therefore associate a point in $\mathbb{P}(\Lambda^2 V)$.

Here $\Lambda^2 V$ corresponds with the $\mathfrak{so}(n)$ from earlier, and not Λ from earlier. I think I've figured it out that there is a correspondence between the space of all alternating bilinear functions and the space $\mathfrak{so}(n)$ of skew symmetric matrices. The correspondence is each B is represented by a matrix $M \in \mathfrak{so}(n)$ such that $M_{ij} = B(v_i, v_j)$ for $\{v_i\}_{1 \leq i \leq n}$ a basis of V. To be precise, let $\mathscr{B}$ be the vector space of all bilinear transformations on V. Then there is an isomorphism $\mathscr{B} \to \mathfrak{so}(n)$ given by $B \mapsto (B(v_i, v_j))$. What is funny is that $B : V \times V \to k$ and for $s \in \mathfrak{so}(n)$, $s : V \to V$. Then we have $u \wedge v$ which takes an input of $B \in \mathscr{B}$ and outputs an element of k, or receives an element of V and outputs an element of V. That is $u \wedge v$ can be thought of as, $u \wedge v : \mathscr{B} \to k$ or $u \wedge v : V \to V$. The isomorphism between $\mathscr{B}$ and $\mathfrak{so}(n)$ helps this make sense. Thus we have

$$\begin{array}{ccccc} & & \{u \wedge v\} & & \\ & \swarrow_{\ncong} & & \searrow^{\ncong} & \\ \mathfrak{so}(n) & \overset{\cong}{\longleftrightarrow} & \mathscr{B} & \overset{\cong}{\longleftrightarrow} & \mathscr{B}^* = \Lambda^2 V \end{array}.$$

s could also act $V \times V \to k$ via $(x, y) \mapsto x^T s y$, which is bilinear. This should correspond with the appropriate B, but I won't check that.

The problem is, not every vector in $\Lambda^2 V$ can be written as $u \wedge v$ for vectors $u, v \in V$. In general it is a linear combination of such expressions. The task, in order to describe the space of lines, is to characterize such *decomposable* 2-vectors.

4.2 Higher exterior powers

Definition 15. An *alternating multilinear form* of degree p on a vector space V is a map $M : V^p \to k$ such that

- For all $\sigma \in S_p$, $M(u_1, \dots, u_p) = \operatorname{sgn}(\sigma) M\left(u_{\sigma(1)}, \dots, u_{\sigma(p)}\right)$;
- $M(\lambda_1 v_1 + \lambda_2 v_2, u_2, \dots, u_p) = \lambda_1 M(v_1, u_2, \dots, u_p) + \lambda_2 M(v_2, u_2, \dots, u_p)$

Example: Let $u_1, \dots, u_n$ be column vectors. Then

$$M(u_1, \dots, u_n) = \det(u_1 u_2 \cdots u_n)$$

is an alternating multilinear form of degree n.

The set of all alternating multilinear forms on V is a vector space, and M is unique minamined by the values

$$M(v_{i_1}, v_{i_2}, \dots, v_{i_p})$$

for a basis $\{v_1, \dots, v_n\}$. But the alternating property allows us to change the order as long as we multiply by -1 for each transposition. This means that M is uniquely determined by the values of indices for

$$i_1 < i_2 < \cdots < i_p.$$

The number of these is the number of p-element subsets of $[n]$, i.e. $\binom{n}{p}$, so this is the dimension of the space of such forms. In particular if $p > n$ this space is zero. We define analogous constructions to those above for a pair of vectors:

Definition 16. The p^{th} *exterior power* $\Lambda^p V$ of a finite dimensional vector space is the dual space of the vector space of alternating multilinear forms of degree p on V. The elements of $\Lambda^p V$ are called p-vectors.

and

Definition 17. Given $u_1, \dots, u_p \in V$, the *exterior product* $u_1 \wedge u_2 \wedge \cdots \wedge u_p \in \Lambda^p V$ is the linear map to F which, on an alternating multilinear form M takes the value

$$(u_1 \wedge \cdots \wedge u_p)(M) = M(u_1, \dots, u_p).$$

The exterior product $u_1 \wedge \cdots \wedge u_p$ has three defining properties:

- it is linear in each variable u_i separately
- interchanging two variables changes the sign of the product
- if two variables are the same the exterior product vanishes.

We have a useful generalization of the earlier proposition:

Proposition 16. The exterior product $u_1 \wedge \cdots \wedge u_p$ of p vectors $u_i \in V$ vanishes if and only if the vectors are linearly dependent.

The exterior powers $\Lambda^p V$ have natural properties with respect to linear transformations: given a linear transformation $T : V \to W$, and an alternating multilinear form M on W, we can define an induced one T^*M on V by

$$T^*M(v_1, \ldots, v_p) = M(Tv_1, \ldots, Tv_p)$$

and this defines a dual linear map

$$\Lambda^p T : \Lambda^p V \to \Lambda^p W$$

with the property that

$$\Lambda^p T(v_1 \wedge \cdots \wedge v_p) = Tv_1 \wedge \cdots \wedge Tv_p.$$

One such map is very familiar: take $p = n$, so that $\Lambda^n V$ is one-dimensional and spanned by $v_1 \wedge \cdots \wedge v_n$ for a basis $\{v_1, \ldots, v_n\}$. A linear transformation from a 1-dimensional vector space to itself is just multiplication by a scalar, so $\Lambda^n T$ is some scalar in the field. In fact it is the *determinant* of T. To see this, observe that

$$\Lambda^n T(v_1 \wedge \cdots \wedge v_n) = Tv_1 \wedge \cdots \wedge Tv_n$$

and the right hand side can be written using the matrix T_{ij} of T as

$$\sum_{i_1,\ldots,i_n} T_{i_1 1} v_{i_1} \wedge \cdots \wedge T_{i_n n} v_{i_n} = \sum_{i_1,\ldots,i_n} T_{i_1 1} \cdots T_{i_n n} v_{1_i} \wedge \cdots \wedge v_{i_n}.$$

Each of the terms vanishes if any two of $i_1, \ldots, i_n$ are equal by the property of the exterior product, so we need only consider the case where $(i_1, \ldots, i_n)$ is a permutation of $(1, \ldots, n)$.

We now have vector spaces $\Lambda^p V$ of dimension $\binom{n}{p}$ naturally associated to V. The space $\Lambda^1 V$ is by definition the dual space of the space of linear functions on V, so $\Lambda^1 V = V^{**} \cong V$ and by convention we set $\Lambda^0 V = k$. Given p vectors $v_1, \ldots, v_p \in V$ we also have a corresponding vector $v_1 \wedge \cdots \wedge v_p \in \Lambda^p V$ and the notation suggests that there should be a product so that we can remove the brackets:

$$(u_1 \wedge \cdots \wedge u_p) \wedge (v_1 \wedge \cdots \wedge v_q) = u_1 \wedge \cdots \wedge u_p \wedge v_1 \wedge \cdots \wedge v_q.$$

And indeed there is. So suppose $a \in \Lambda^p V, b \in \Lambda^q V$, we want to define $a \wedge b \in \Lambda^{p+q} V$. Now for fixed vectors $u_1, \ldots, u_p \in V$,

$$M(u_1, \ldots, u_p, v_1, \ldots, v_q)$$

is an alternating q-linear function of $v_1, \ldots, v_q$, so if

$$b = \sum_{1 \le j_1 < \cdots < j_q \le p+q} \lambda_{j_1 \cdots j_q} v_{j_1} \wedge \cdots \wedge v_{j_q}$$

then

$$\sum_{j_1 < \cdots < j_q} \lambda_{j_1 \cdots j_q} M(u_1, \ldots, u_p, v_{j_1}, \ldots, v_{j_q})$$

only depends on b and not on the particular way it is written in terms of a basis $\{v_1 \ldots, v_n\}$. Similarly, if

$$a = \sum_{1 \le i_1 < \cdots < i_p \le p+q} \mu_{i_1 \cdots i_p} u_{i_1} \wedge \cdots \wedge u_{i_p}$$

then

$$\sum_{i_1 < \cdots < i_p} \mu_{i_1 \cdots i_p} M(u_{i_1}, \ldots, u_{i_p}, v_1, \ldots, v_q)$$

depends on a. We can therefore unambiguously define $a \wedge b$ by its value on an alternating $p+q$-form M as

$$(a \wedge b)(M) = \sum_{i_1 < \cdots < i_p; j_1 < \cdots < j_q} \mu_{i_1 \cdots i_p} \lambda_{j_1 \cdots j_q} M(u_{i_1}, \ldots, u_{i_p}, v_{j_1}, \ldots, v_{j_q}).$$

The product just involves linearity and removing the brackets.

Example 1. Suppose $a = v_1 + v_2$, $b = v_1 \wedge v_3 - v_3 \wedge v_2$, with $v_1, v_2, v_3 \in V$. Then

$$\begin{aligned} a \wedge b &= (v_1 + v_2) \wedge (v_1 \wedge v_3 - v_3 \wedge v_2) \\ &= v_1 \wedge v_1 \wedge v_3 - v_1 \wedge v_3 \wedge v_2 + v_2 \wedge v_1 \wedge v_3 - v_2 \wedge v_3 \wedge v_2 \\ &= -v_1 \wedge v_3 \wedge v_2 + v_2 \wedge v_1 \wedge v_3 \\ &= v_1 \wedge v_2 \wedge v_3 - v_1 \wedge v_2 \wedge v_3 = 0 \end{aligned}$$

where we have used the basic rules that a repeated vector from V in an exterior product gives zero, and the transposition of two vectors changes the sign.

Note that

$$u_1 \wedge \cdots \wedge u_p \wedge v_1 \wedge \cdots \wedge v_q = (-1)^p v_1 \wedge u_1 \wedge \cdots \wedge u_p \wedge v_2 \wedge \cdots \wedge v_q$$

because we have to interchange v_1 with each of the p u_i's to bring it to the front, and then repeating

$$u_1 \wedge \cdots \wedge u_p \wedge v_1 \wedge \cdots \wedge v_q = (-1)^{pq} v_1 \wedge \cdots \wedge v_q \wedge u_1 \wedge \cdots \wedge u_p.$$

This extends by linearity to all $a \in \Lambda^p V, b \in \Lambda^q V$. We then have the basic properties of the exterior product:

- $a \wedge (b + c) = a \wedge b + a \wedge c$
- $(a \wedge b) \wedge c = a \wedge (b \wedge c)$
- $a \wedge b = (-1)^{pq} b \wedge a$ if $a \in \Lambda^p V, b \in \Lambda^q V$.

What we have done may seem rather formal, but it has many concrete applications. For example, if $a = x \wedge y$, then $a \wedge a = x \wedge y \wedge x \wedge y = 0$ because $x \in V = \Lambda^1 V$ is repeated. So it is much easier to determine that $a = v_1 \wedge v_2 + v_3 \wedge v_4$ ($v_1, v_2, v_3, v_4 \in V$ linearly independent) is not decomposable:

$$(v_1 \wedge v_2 + v_3 \wedge v_4) \wedge (v_1 \wedge v_2 + v_3 \wedge v_4) = 2 v_1 \wedge v_2 \wedge v_3 \wedge v_4 \neq 0.$$

4.3: Decomposable 2-vectors

A line in $\mathbb{P}(V)$ defines a point in $\mathbb{P}(\Lambda^2 V)$ defined by a *decomposable* 2-vector

$$a = x \wedge y.$$

We need to characterize algebraically this decomposability, and the following theorem does just that:

Theorem 17. Let $a \in \Lambda^2 V$ be a non-zero element. Then a is decomposable if and only if $a \wedge a = 0 \in \Lambda^4 V$.

Proof. ($\Rightarrow$) If $a = x \wedge y$ for two vectors x and y, then

$$a \wedge a = x \wedge y \wedge x \wedge y = 0$$

because of the repeated factor x (or y).

($\Leftarrow$) We prove the converse by induction on the dimension of V. If $\dim V = 0, 1$ then $\Lambda^2 V = 0$, so the first case is $\dim V = 2$. In this case $\dim \Lambda^2 V = 1$ and $v_1 \wedge v_2$ is a nonzero element if $\{v_1, v_2\}$ is a basis for V, so any a is decomposable.

We consider the case $\dim V = 3$ separately now. Given a non-zero $a \in \Lambda^2 V$, define $A : V \to \Lambda^3 V$ by

$$A(v) = a \wedge v.$$

Since $\dim \Lambda^3 V = 1$, $\dim \ker A \geq 2$, so let u_1, u_2 be linearly independent vectors in the kernel and extend to a basis u_1, u_2, u_3 of V. Then write

$$a = \lambda_1 u_2 \wedge u_3 + \lambda_2 u_3 \wedge u_1 + \lambda_3 u_1 \wedge u_2.$$

Now by definition $0 = a \wedge u_1 = \lambda_1 u_2 \wedge u_3 \wedge u_1$ so $\lambda_1 = 0$ and similarly $0 = a \wedge u_2$ implies $\lambda_2 = 0$. It follows that $a = \lambda_3 u_1 \wedge u_2$, which is decomposable.

Now assume inductively that the theorem is true for $\dim V \leq n-1$ and consider the case $\dim V = n$. Using a basis $v_1, \ldots, v_n$, write

$$\begin{aligned} a &= \sum_{1 \leq i < j}^{n} a_{ij} v_i \wedge v_j \\ &= \left(\sum_{i=1}^{n-1} a_{in} v_i \right) \wedge v_n + \sum_{1 \leq i < j}^{n-1} a_{ij} v_i \wedge v_j \\ &= u \wedge v_n + a' \end{aligned}$$

where $u \in U = k\langle v_1, \ldots, v_{n-1} \rangle$ and $a' \in \Lambda^2 U$.

Now

$$0 = a \wedge a = (u \wedge v_n + a') \wedge (u \wedge v_n + a') = 2u \wedge a' \wedge v_n + a' \wedge a'.$$

But v_n doesn't appear in the expansion of $u \wedge a'$ or $a' \wedge a'$, so we separately obtain

$$u \wedge a' = 0, \qquad\qquad a' \wedge a' = 0.$$

By induction $a' \wedge a' = 0$ implies $a' = u_1 \wedge u_2$ and so the earlier equation reads

$$u \wedge u_1 \wedge u_2 = 0$$

which means that there is a linear relation

$$\lambda u + \mu_1 u_1 + \mu_2 u_2 = 0.$$

If $\lambda = 0$, then u_1 and u_2 are linearly dependent so $a' = u_1 \wedge u_2 = 0$. This means that $a = u \wedge v_n$ and is therefore decomposable. If $\lambda \neq 0$, $u = \lambda_1 u_1 + \lambda_2 u_2$, so

$$a = \lambda_1 u_1 \wedge v_n + \lambda_2 u_2 \wedge v_n + u_1 \wedge u_2$$

and this is the 3-dimensional case which is always decomposable as shown above. We conclude that a in each case is decomposable. □

4.4 The Klein quadric:

The first case where we can apply Theorem 17 is when $\dim V = 4$, to describe the projective lines in the 3-dimensional space $\mathbb{P}(V)$. In this case $\dim \Lambda^4 V = 1$ with a basis vector $v_0 \wedge v_1 \wedge v_2 \wedge v_3$ if V is given the basis $v_0, \dots, v_3$.

For $a \in \Lambda^2 V$ we write

$$a = \lambda_1 v_0 \wedge v_1 + \lambda_2 v_0 \wedge v_2 + \lambda_3 v_0 \wedge v_3 + \mu_1 v_2 \wedge v_3 + \mu_2 v_3 \wedge v_1 + \mu_3 v_1 \wedge v_2$$

and then $a \wedge a = B(a, a) v_0 \wedge v_1 \wedge v_2 \wedge v_3$ where

$$B(a, a) = 2(\lambda_1\mu_1 + \lambda_2\mu_2 + \lambda_3\mu_3). \tag{2}$$

This is a non-degenerate quadratic form, and so $B(a, a) = 0$ defines a nonsingular quadric $Q \subseteq \mathbb{P}(\Lambda^2 V)$. Moreover, any other choice of basis rescales B by a non-zero constant and so Q is well-defined in projective space.

We see then that a line $\ell \subseteq \mathbb{P}(V)$ defines a decomposable 2-vector $a = x \wedge y$, unique up to a scalar and since $a \wedge a = 0$, it defines a point $L \in Q \subseteq \mathbb{P}(\Lambda^2 V)$. Conversely, Theorem 17 tells us that every point in Q is represented by a decomposable 2-vector. Hence

Proposition 18. There is a one-to-one correspondence $\ell \longleftrightarrow L$ between lines ℓ in a 3-dimensional projective space $\mathbb{P}(V)$ and points L in the 4-dimensional quadric $Q \subseteq \mathbb{P}(\Lambda^2 V)$.

It was Felix Klein, building on the work of his supervisor Julius Plücker, who first described this in detail and Q is usually called the *Klein quadric*. The equation of the quadric in the form (2) shows that there are linear subspaces inside it of maximal dimension 2 whatever the field. The linear subspaces all relate to intersection properties of lines in $\mathbb{P}(V)$. For example:

Proposition 19. Two lines $\ell_1, \ell_2 \subseteq \mathbb{P}(V)$ intersect if and only if the line joining the two corresponding points $L_1, L_2 \in Q$ lies entirely in Q.

Proof. ($\Rightarrow$) Let $U_1, U_2 \subseteq V$ be the two dimensional subspaces of V defined by ℓ_1, ℓ_2. Suppose the lines intersect in X, with representative vector $u \in V$. Then extend to bases $\{u, u_1\}$ for U_1 and $\{u, u_2\}$ for U_2. The line in $\mathbb{P}(\Lambda^2 V)$ joining L_1 and L_2 is then $\mathbb{P}(W)$ where W is spanned by $u \wedge u_1$ and $u \wedge u_2$.

Any 2-vector in W is thus of the form

$$\lambda_1 u \wedge u_1 + \lambda_2 u \wedge u_2 = u \wedge (\lambda_1 u_1 + \lambda_2 u_2)$$

which is decomposable and so represents a point in Q.

($\Leftarrow$) Conversely, if the lines do not intersect, $U_1 \cap U_2 = \{0\}$, so $V = U_1 \oplus U_2$. In this case choose bases $\{u_1, v_1\}$ of U_1 and $\{u_2, v_2\}$ of U_2. Then $\{u_1, v_1, u_2, v_2\}$ is a basis of V and in particular $u_1 \wedge v_1 \wedge u_2 \wedge v_2 \neq 0$. A point on the line joining L_1 and L_2 is now represented by $a = \lambda_1 u_1 \wedge v_1 + \lambda_2 u_2 \wedge v_2$ so that

$$a \wedge a = 2\lambda_1 \lambda_2 u_1 \wedge v_1 \wedge u_2 \wedge v_2$$

which vanishes only if λ_1 or λ_2 are zero. Thus the line only meets Q at the points L_1 and L_2. □

Now fix a point $X \in \mathbb{P}(V)$ and look at the set of lines passing through this point:

Proposition 20. The set of lines $\ell \subseteq \mathbb{P}(V)$ passing through a fixed point $X \in \mathbb{P}(V)$ corresponds to the set of points $L \in Q$ such lie in a fixed plane contained in Q.

Proof. Let x be a representative vector for X. The line $\mathbb{P}(U)$ passes through X if and only if $x \in U$, so $\mathbb{P}(U)$ is represented in the Klein quadric by a 2-vector of the form

$$x \wedge u.$$

Extend x to a basis $\{x, v_1, v_2, v_3\}$ of V, then any decomposable 2-vector of the form $x \wedge y$ can be written as

$$x \wedge (\mu x + \lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3) = \lambda_1 x \wedge v_1 + \lambda_2 x \wedge v_2 + \lambda_3 x \wedge v_3.$$

Thus any line passing through X is represented by a 2-vector in the 3-dimensional space of decomposables spanned by $x \wedge v_1, x \wedge v_2, x \wedge v_3$, which is a projective plane in Q. Conversely any point in this plane defines a line in $\mathbb{P}(V)$ through X. □

A plane in Q defined by a point $X \in \mathbb{P}(V)$ like this is called an α-plane. There are other planes in Q:

Proposition 21. Let $\mathbb{P}(W) \subseteq \mathbb{P}(V)$ be a plane. The set of lines $\ell \subseteq \mathbb{P}(W)$ corresponds to the set of points $L \in Q$ which lie in a fixed plane contained in Q.

A plane of this type contained in Q is called a β-plane.

Proof. We just use duality here: if $U \subseteq V$ is 2-dimensional, then its annihilator $U^0 \subseteq V^*$ is $4 - 2 = 2$-dimensional, so there is a one-to-one correspondence between lines in $\mathbb{P}(V)$ and lines in $\mathbb{P}(V^*)$. A point in Q therefore defines a line in either the projective space or its dual. Now the dual of the set of lines passing through a point is the set of lines lying in a (hyper)-plane. So applying Proposition 20 to $\mathbb{P}(V^*)$ gives the result. □

In fact, there are no more planes:

Proposition 22. Any plane in the Klein quadric Q is either an α-plane or a β-plane.

Proof. Take a plane in Q and three non-collinear points L_1, L_2, L_3 on it. We get three lines $\ell_1, \ell_2, \ell_3 \subseteq \mathbb{P}(V)$. Since the line joining L_1 to L_2 lies in the plane and hence in Q, it follows from Proposition 19 that each pair of ℓ_1, ℓ_2, ℓ_3 intersect. There are two possibilities:

- the three lines form a pencil;
- the three lines meet at three distinct points.

In the first case the three lines pass through a single point and so L_1, L_2, L_3 lie in an α-plane. But this must be the original plane since the three representative vectors for L_1, L_2, L_3 are linearly independent as the points are not collinear.

In the second case, if u_1, u_2, u_3 are representative vectors for the three points of intersection of ℓ_1, ℓ_2, ℓ_3, then L_1, L_2, L_3 are represented by $u_2 \wedge u_3, u_3 \wedge u_1, u_1 \wedge u_2$. A general point on the plane is then given by

$$\lambda_1 u_2 \wedge u_3 + \lambda_2 u_3 \wedge u_1 + \lambda_3 u_1 \wedge u_2$$

which is a general element of $\Lambda^2 U$ where U is spanned by u_1, u_2, u_3. Thus ℓ_1, ℓ_2, ℓ_3 all lie in the plane $\mathbb{P}(U) \subseteq \mathbb{P}(V)$. □

The existence of these two families of linear subspaces of maximal dimension is characteristic of even-dimensional quadrics − it is the generalization of the two familites of lines on the "cooling tower" quadric surface. In the case of the Klein quadric, two different α-planes intersect in a point, since there is a unique line joining two points. Similarly (and by duality) two β planes meet in a point. An α-plane and a β-plane in general have empty intersection: if X is a point and π a plane with $X \notin \pi$, there is no line in π with passes through X. If $X \in \pi$, then the intersection is a line.

Plücker Embedding

The Plücker embedding over the field k is the map ι defined by

$$\begin{aligned} \iota : \mathfrak{Gr}(r, k^n) &\to \mathbb{P}(\Lambda^r k^n) \\ k\langle v_1, \dots, v_r\rangle &\mapsto [v_1 \wedge \cdots \wedge v_r], \end{aligned}$$

where $\mathfrak{Gr}(r, k^n)$ is the Grassmannian, i.e., the space of all r-dimensional subspaces of the n-dimensional vector space k^n.

This is an isomrophism from the Grassmannian to the image of ι, which is a projective variety. This variety can be completely characterized as an intersection of quadrics, each coming from a relation on the Plücker (or Grassmann) coordinates that derives from linear algebra.

The embedding of the Grassmannian satisfies some very simple quadratic relations called the *Plücker relations*. These show that the Grassmannian embeds as an algebraic subvariety of $\mathbb{P}(\Lambda^r V)$ and give another method of constructing the Grassmannian. To state the

Plücker relations, let W be the r-dimensional subspace spanned by the basis of row vectors $\{w_1, \ldots, w_r\}$. Let U be the $r \times n$ matrix of homogeneous coordinates whose rows are $\{w_1, \ldots, w_r\}$ and let $\{W_1, \ldots, W_n\}$ be the corresponding column vectors. For any ordered sequence $1 \leq i_1 < \cdots < i_r \leq n$ of positive integers, let $W_{i_1,\ldots,i_r}$ be the determinant of the $r \times r$ minor with the columns $(W_{i_1}, \ldots, W_{i_r})$. Then $\{W_{i_1,\ldots,i_r}\}$ are the Plücker coordinates of the element U of the Grassmannian. They are the linear coordinates of the image $\iota(W)$ of W under the Plücker map, relative to the standard basis in the exterior power $\Lambda^r V$.

For any two ordered sequences $i_1 < \cdots < i_{r-1}$ and $j_1 < \cdots < j_{r+1}$ of positive integers $1 \leq i_\ell, j_m \leq n$, the following homogeneous quadratics determine the image of $\mathfrak{Gr}(r, k^n)$ under the Plücker map:

$$\sum_{\ell=1}^{r+1} (-1)^\ell W_{i_1,\ldots,i_{r-1},j_\ell} W_{j_1,\ldots,\hat{j_\ell},\ldots,j_{r+1}} = 0.$$

For example, when $\dim V = 4$ and $r = 2$, we have the Klein quadric determined by the equation $W_{12}W_{34} - W_{13}W_{24} + W_{14}W_{23}$. In general, however, many more equations are needed to carve the image of the Grassmannian under the Plücker embedding.

Note $\dim \mathfrak{Gr}(r, k^n) = r(n-r) < \binom{n}{r} - 1 = \dim \mathbb{P}(\Lambda^r k^n)$.