Definition 1.1. A **Steiner triple system** is an ordered pair (S,T) where S is a finite set of points or symbols, and T is a set of 3-element subsets of S called triples, such that each pair of distinct elements of S occurs together in exactly one triple of T.

One example is the Fano plane. An equivalent way of thinking of a Steiner triple system is as a partitioning of the edges of the complete graph $K_{|S|}$ into triangles.

Theorem 1.1.3. A Steiner triple system of order v exists if and only if $v \equiv 1$ or $3 \mod 6$.

Proof. If (S,T) is a Steiner triple system of order v, the triple $\{a,b,c\}$ contains exactly three two-element subsets: $\{a,b\}$, $\{a,c\}$, and $\{b,c\}$. Note that S itself contains $\binom{v}{2}$-two element subsets. Since every pair much exist within one triple in T, we must have

$$3|T| = \binom{v}{2} = \frac{v(v-1)}{2}.$$

And so

$$|T| = \frac{v(v-1)}{6}.$$

For any $x \in S$, define the set $T(x) = \{t \setminus \{x\} : x \in t \in T\}$. Then $T(x)$ partitions $S \setminus \{x\}$ into two-element subsets. This is because every $s \in S \setminus \{x\}$ is in exactly one triple alongside x, so if $y \in S$ is such that $\{x,y,s\} \in T$, then $\{y,s\}$ is the only pair in $T(x)$ containing s.

Thus $v-1$ is even, so v must be odd. Therefore $v \equiv 1$ or 3 or $5 \mod 6$. However, $\frac{v(v-1)}{6}$ is never an integer for $v \equiv 5 \mod 6$. This is because $5(5-1) = 20 \equiv 2 \mod 6 \not\equiv 0 \mod 6$. So we are left with $v \equiv 1$ or $3 \mod 6$. □

It remains to show there exists a Steiner system for every such number. We will see this with various construction methods.

Exercise 1.1.4. Let S be a finite set of size v and let T be a set of triples of S satisfying

1. each pair of distinct elements of S belongs to *at least* one triple in T, and
2. $|T| \leq v(v-1)/6$.

Show that (S,T) is a Steiner triple system.

Proof. Since $|T| \leq v(v-1)/6$, we know that $3|T| \leq \binom{v}{2}$. For each pair $\{x,y\} \in \mathcal{P}(S,2)$, denote by $N_{\{x,y\}}$ the number of triples $t \in T$ that satisfy $\{x,y\} \subseteq t$. Let N be the sum total of the number of times each pair is present in a triple of T. Because of property (a), $N \geq \binom{v}{2}$. But because each triple always contains exactly three pairs, we have $N = 3|T|$. Therefore $N = 3|T| = \binom{v}{2}$.

Suppose there existed $z \neq w \in S$ such that $\{x,y,z\} = \{x,y,w\}$. Then $N_{\{x,y\}} > 1$. But then $N > \binom{v}{2}$, which contradicts our earlier result. So we must have $N_{\{x,y\}} = 1$, and so we have a Steiner triple system.

□

Definition 1.2. A **latin square** of order n is an $n \times n$ array, each cell of which contains exactly one of the numbers $\{1, \ldots, n\}$ and each column and row of which contains exactly one of each of the numbers $\{1, \ldots, n\}$. A **quasigroup** of order n is a pair $(Q, \circ)$ where Q is a set of n elements and $\cdot \circ \cdot : Q \times Q \to Q$ is a binary operation such that for every pair of elements $a, b \in Q$ the equations $a \circ x = b$ and $y \circ a = b$ have unique solutions.

A latin square and a quasigroup are essentially the same thing. The former is merely the multiplication table of the latter.

The Bose Construction ($v \equiv 3 \mod 6$): Let $v = 6n+3$ and let Q be an idempotent quasigroup of order $2n+1$. Let $S = Q \times \{a, b, c\}$ and define T to contain the following two types:

1. For $1 \leq i \leq 2n+1$, $\{(i, a), (i, b), (i, c)\} \in T$.
2. For $1 \leq i < j \leq 2n+1$ $\{(i, a), (j, a), (i \circ j, b)\}, \{(i, b), (j, b), (i \circ j, c)\}, \{(i, c), (j, c), (i \circ j, a)\} \in T$.

Then (S, T) is a Steiner triple system. This can be proven by applying Exercise 1.1.4.

Proof. First we will show that $|T| \leq v(v-1)/6$. Recall now that $v = 6n+3$. The triples of Type 1 can be counted as $2n+1$. The triples of Type 2 can be counted as $3(2n+1)(2n)/2 = 3(2n^2+n) = 6n^2+3n$. Therefore $|T| = 6n^2+5n+1$. Note here that $v(v-1)/6 = (6n+3)(6n+2)/6 = (36n^2+30n+6)/6 = 6n^2+5n+1$. So in fact $|T| = v(v-1)/6$.

Next we need to demonstrate that every pair is in at least one triple of T. Let $\{(i, p), (j, q)\}$ be a pair, where $i, j \in Q$ and $p, q \in \{a, b, c\}$. First suppose $i = j$. Then $\{(i, p), (j, q)\} \subseteq \{(i, a), (i, b), (i, c)\}$, the Type 1 triple.

Now suppose $p = q$. Then $\{(i, p), (j, q)\} \subseteq \{(i, p), (j, p), (i \circ j, p+1)\}$, the Type 2 triple.

Finally, suppose $i \neq j$ and $p \neq q$. Then either $p = q+1$ or $q = p+1$. First suppose $q = p+1$. Then we want to show there is a $k \in Q$ such that $\{(i, p), (k, p), (j, q = p+1)\} \in T$. The element k must satisfy $i \circ k = j$. Because Q is a quasigroup, this equation has a unique solution k, so we are done. NOTE: it is impossible that $k = i$ because Q is idempotent.

The same thing goes for the case $p = q+1$. Therefore every pair is in at least one triple in T. By Exercise 1.1.4, (S, T) is thus a Steiner triple system. □

Example JK1. Let us construct a Steiner triple system of nine elements. First let us take the quasigroup

$Q =$

$\circ$	1	2	3
1	1	3	2
2	3	2	1
3	2	1	3

Then the Bose Steiner triple system $(Q \times \{a, b, c\}, T)$ contains $|T| = 12$ triples.

$$\{(1,a),(1,b),(1,c)\},\{(2,a),(2,b),(2,c)\},\{(3,a),(3,b),(3,c)\},$$
$$\{(1,a),(2,a),(3,b)\},\{(1,b),(2,b),(3,c)\},\{(1,c),(2,c),(3,a)\},$$
$$\{(1,a),(3,a),(2,b)\},\{(1,b),(3,b),(2,c)\},\{(1,c),(3,c),(2,a)\},$$
$$\{(2,a),(3,a),(1,b)\},\{(2,b),(3,b),(1,c)\},\{(2,c),(3,c),(1,a)\}.$$

This can be expressed in Grünbaum's configuration notation as a $(9_4, 12_3)$, because there are 9 points, 4 "lines" per point, 12 "lines" and 3 points per "line." This is the same kind of configuration as that formed by the nine flex points of an elliptic curve.

So the Bose method can form a $([6n+3]_{[3n+1]}, [6n^2+5n+1]_3)$-configuration. So we can make a $(15_7, 35_3)$-configuration, a $(21_{10}, 70_3)$-configuration, a $(27_{13}, 117_3)$-configuration, and a $(33_{16}, 176_3)$-configuration etc. An $(81_{40}, 1080_3)$-configuration etc.

Example JK2. Let us construct a Steiner triple system of fifteen elements. First let us take the quasigroup

$Q =$

$\circ$	1	2	3	4	5
1	1	5	4	3	2
2	5	4	3	2	1
3	4	3	2	1	5
4	3	2	1	5	4
5	2	1	5	4	3

Note that this quasigroup is commutative but not idempotent! Does it fail to produce a Steiner triple system? Then the Bose construction yields $(Q \times \{a, b, c\}, T)$ contains $|T| = 35$ triples. The five Type 1 triples:

$$\{(1,a),(1,b),(1,c)\},$$
$$\{(2,a),(2,b),(2,c)\},$$
$$\{(3,a),(3,b),(3,c)\},$$
$$\{(4,a),(4,b),(4,c)\},$$
$$\{(5,a),(5,b),(5,c)\},$$

and the thirty Type 2 triples:

$$\begin{gathered}\{(1,a),(2,a),(5,b)\},\{(1,b),(2,b),(5,c)\},\{(1,c),(2,c),(5,a)\},\\ \{(1,a),(3,a),(4,b)\},\{(1,b),(3,b),(4,c)\},\{(1,c),(3,c),(4,a)\},\\ \{(1,a),(4,a),(3,b)\},\{(1,b),(4,b),(3,c)\},\{(1,c),(4,c),(3,a)\},\\ \{(1,a),(5,a),(2,b)\},\{(1,b),(5,b),(2,c)\},\{(1,c),(5,c),(2,a)\},\\ \{(2,a),(3,a),(3,b)\},\{(2,b),(3,b),(3,c)\},\{(2,c),(3,c),(3,a)\},\\ \{(2,a),(4,a),(2,b)\},\{(2,b),(4,b),(2,c)\},\{(2,c),(4,c),(2,a)\},\\ \{(2,a),(5,a),(1,b)\},\{(2,b),(5,b),(1,c)\},\{(2,c),(5,c),(1,a)\},\\ \{(3,a),(4,a),(1,b)\},\{(3,b),(4,b),(1,c)\},\{(3,c),(4,c),(1,a)\},\\ \{(3,a),(5,a),(5,b)\},\{(3,b),(5,b),(5,c)\},\{(3,c),(5,c),(5,a)\},\\ \{(4,a),(5,a),(4,b)\},\{(4,b),(5,b),(4,c)\},\{(4,c),(5,c),(4,a)\}.\end{gathered}$$

This fails to be a Steiner triple system because for example the pair $\{(5,a),(5,b)\}$ appears in the triple $\{(5,a),(5,b),(5,c)\}$ of Type 1 and the triple $\{(3,a),(5,a),(5,b)\}$ of Type 2. This ultimately results from the non-idempotent nature of $3 \circ 5 = 5$.

Definition 1.3.1. A quasigroup of even order $2n$ is **half-idempotent** if $i \circ i = i$ for all $i \leq n/2$ and $i \circ i = i - n/2$ for all $i > n/2$.

Note that for every odd n, you can make a commutative idempotent quasigroup out of $\mathbb{Z}/n\mathbb{Z}$ by $i \circ j = \frac{i+j}{2}$, since $2 \in (\mathbb{Z}/n\mathbb{Z})^\times$. For even $2n$, you can partition $\mathbb{Z}/2n\mathbb{Z}$ into $A = \{0, \ldots, n-1\}$ and $B = \{n, \ldots, 2n-1 = -1\}$. Then for the equation $y = 2x$, denote by A_y the solution x in the set A, and denote by B_y the solution x in the set B. Then

$$i \circ j = \begin{cases} A_{i+j} & i+j \text{ is even} \\ B_{i+j-1} & i+j \text{ is odd} \end{cases}$$

is a half-idempotent commutative quasigroup.

The Skolem Construction ($v \equiv 1 \mod 6$): Let Q be a half-idempotent commutative quasigroup of order $2n$, where $Q = \{1, \ldots, 2n\}$. Define $S = \{\infty\} \cup (Q \times \{a,b,c\})$. Define T as following three types:

1. For $1 \leq i \leq n$, $\{(i,a),(i,b),(i,c)\} \in T$.
2. For $1 \leq i \leq n$, $\{\infty,(n+i,a),(i,b)\},\{\infty,(n+i,b),(i,c)\},\{\infty,(n+i,c),(i,a)\} \in T$.
3. For $1 \leq i < j \leq 2n$, $\{(i,a),(j,a),(i \circ j,b)\},\{(i,b),(j,b),(i \circ j,c)\},\{(i,c),(j,c),(i \circ j,a)\} \in T$.

Then (S,T) is a Steiner triple system.

Proof. Let us again turn to Exercise 1.1.4. Counting up the triples of T, we get n triples of Type 1, $3n$ triples of Type 2, and $3 * 2n(2n-1)/2 = 6n^2 - 3n$ triples of Type 3. Adding these up, we get $|T| = 6n^2 + n$. Note that with $v = 6n+1$, we have $v(v-1)/6 = (6n+1)(6n)/6 = 6n^2 + n$. And so $|T| = v(v-1)/6$.

Next, we must show that every pair is present in at least one triple of T.

- First let us consider pairs of the form $\{(i,p),(j,q)\}$ where $i,j \in Q$ and $p,q \in \{a,b,c\}$. If $i = j \leq n$, then $\{(i,p),(j,q)\}$ is in a triple of Type 1.

- If $i = j > n$, then write $i = j = k + n$ for some $k \in Q$. So we wish to find a triple of T containing $\{(k+n,p),(k+n,q)\}$. Suppose that $q = p+1$. Then we wish to find an $\ell \in Q$ such that $\{(\ell,p),(k+n,p),(k+n,q=p+1)\} \in T$. This is a Type 3 triple. This is true if $\ell \circ (k+n) = k+n$. Because Q is a quasigroup, ℓ exists and is unique. So the pair $\{(i,p),(j,q)\}$ is in at least one triple. The same goes for if $p = q+1$. (NOTE: importantly, $\ell \neq k+n$ because in a half-idempotent quasigroup $(k+n)\circ(k+n) = k \neq k+n$.)

- Now suppose $p = q$. Then $\{(i,p),(j,p)\}$ is in the Type 3 triple $\{(i,p),(j,p),(i\circ j, p{+}1)\}$.

- Now suppose $i \neq j$ and $p \neq q$. Suppose $j = n+i$ and $p = q+1$. then $\{(i,p),(j,q)\}$ is in the Type 2 triple $\{\infty,(j,q),(i,p)\}$.

- Now suppose $j = n+i$ and $q = p+1$. Then $\{(i,p),(j,q)\}$ is in the Type 3 triple $\{(i,p),(x,p),(j,q)\}$ where x solves $i \circ x = j$. Note it is impossible that $x = i$ because $i \circ i = i$ since $i \leq n$ and Q is half-idempotent.

- Now suppose $i \neq j$ do not satisfy $|i-j| = n$ and $p \neq q$. Suppose $q = p+1$. Then we want to find an $x \in Q$ such that $\{(i,p),(x,p),(j,q)\}$ is a Type 3 triple. Then $i \circ x = j$. It is impossible that $x = i$, because $i - (i \circ i) = \begin{cases} 0 & i \leq n \\ n & i > n \end{cases}$. The former case contradicts $i \neq j$ and the latter case contradicts $|i-j| \neq n$. Thus x truly provides us with a Type 3 triple.

- Finally, consider the pair $\{\infty,(i,p)\}$. If $i \leq n$, then $\{\infty,(i,p),(n+i,p-1)\}$ is a Type 2 triple and if $i > n$ then $\{\infty,(i,p),(i-n,p+1)\}$ is a Type 2 triple.

Those are all the possible pairs!!! They are all in some kind of triple in T, so by Exercise 1.1.4, we are done!!! $\square$

So this is a method of constructing a $([6n+1]_{[3n]},[6n^2+n]_3)$-configuration, I suppose. We can make a $(7_3,7_3)$, a $(13_6,26_3)$, $(19_9,57_3)$, a $(25_{12},100_3)$ etc.

Example JK3. Consider the half-idempotent commutative quasigroup Q represented by the matrix $\begin{pmatrix} 1 & 4 & 2 & 3 \\ 4 & 2 & 3 & 1 \\ 2 & 3 & 1 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}$. Then we can construct the $(13_6,26_3)$ Steiner triple system on $S = \{\infty\} \cup (Q \times \{a,b,c\})$ has two triples of Type 1:

$$\{(1,a),(1,b),(1,c)\},\{(2,a),(2,b),(2,c)\},$$

six triples of Type 2:

$$\{\infty,(3,a),(1,b)\},\{\infty,(3,b),(1,c)\},\{\infty,(3,c),(1,a)\},$$
$$\{\infty,(4,a),(2,b)\},\{\infty,(4,b),(2,c)\},\{\infty,(4,c),(2,a)\},$$

and eighteen triples of Type 3:

$$\begin{aligned}
&\{(1,a),(2,a),(4,b)\},\{(1,b),(2,b),(4,c)\},\{(1,c),(2,c),(4,a)\},\\
&\{(1,a),(3,a),(2,b)\},\{(1,b),(3,b),(2,c)\},\{(1,c),(3,c),(2,a)\},\\
&\{(1,a),(4,a),(3,b)\},\{(1,b),(4,b),(3,c)\},\{(1,c),(4,c),(3,a)\},\\
&\{(2,a),(3,a),(3,b)\},\{(2,b),(3,b),(3,c)\},\{(2,c),(3,c),(3,a)\},\\
&\{(2,a),(4,a),(1,b)\},\{(2,b),(4,b),(1,c)\},\{(2,c),(4,c),(1,a)\},\\
&\{(3,a),(4,a),(4,b)\},\{(3,b),(4,b),(4,c)\},\{(3,c),(4,c),(4,a)\},
\end{aligned}$$

Pairwise Balanced Designs There is no Steiner triple system of order $6n+5$, but we can generalize the concept as a **pairwise balanced design** to approximate.

Definition 1.4.1. A **pairwise balanced design** is a set of elements S together with set of blocks $B \subseteq 2^S$ such that every pair of elements of S is in exactly one block in B. (That is, it is like a Steiner triple system without the requirement that all the blocks are size 3, or indeed all the same size at all!)

Example 1.4.1. $S = \{1, 2, \ldots, 11\}$ and B contains the following 16 blocks:

$$\begin{aligned}
\{1,2,3,4,5\}\\
\{1,6,7\}\\
\{1,8,9\}\\
\{1,10,11\}\\
\{2,6,9\}\\
\{2,7,11\}\\
\{2,8,10\}\\
\{3,6,11\}\\
\{3,7,8\}\\
\{3,9,10\}\\
\{4,6,10\}\\
\{4,7,9\}\\
\{4,8,11\}\\
\{5,6,8\}\\
\{5,7,10\}\\
\{5,9,11\}
\end{aligned}$$

Construction not recorded. **Quasigroups with holes and Steiner triple systems:**

Let $Q = \{1, 2, \ldots, 2n\}$ and let $H = \{\{1, 2\}, \{3, 4\}, \ldots, \{2n-1, 2n\}\}$. In what follows, the two-element subsets $\{2i-1, 2i\}$ are called **holes**. A quasigroup Q with holes H is a quasigroup $(Q, \circ)$ of order $2n$ in which for each $h \in H$, $(h, \circ)$ is a subquasigroup of $(Q, \circ)$.

$$\begin{pmatrix} \mathbf{1} & \mathbf{2} & 5 & 6 & 7 & 8 & 3 & 4 \\ \mathbf{2} & \mathbf{1} & 8 & 7 & 3 & 4 & 6 & 5 \\ 5 & 8 & \mathbf{3} & \mathbf{4} & 2 & 7 & 1 & 6 \\ 6 & 7 & \mathbf{4} & \mathbf{3} & 8 & 1 & 5 & 2 \\ 7 & 3 & 2 & 8 & \mathbf{5} & \mathbf{6} & 4 & 1 \\ 8 & 4 & 7 & 1 & \mathbf{6} & \mathbf{5} & 2 & 3 \\ 3 & 6 & 1 & 5 & 4 & 2 & \mathbf{7} & \mathbf{8} \\ 4 & 5 & 6 & 2 & 1 & 3 & \mathbf{8} & \mathbf{7} \end{pmatrix}$$

Exercise 1.5.10. Let $(\{1, 2\}, \circ_1)$ be any quasigroup of order 2 (there are two of them), and let $(Q, \circ_2)$ be an idempotent quasigroup of order $2n+1$ (for example, $a \circ b = \frac{a+b}{2} \mod 2n+1$, which always exists since 2 is a unit mod $2n + 1$).

Let $S = \{1, 2\} \times Q$. Define a binary operation on S by $(a, b) \otimes (c, d) = (a \circ_1 c, b \circ_2 d)$. Then $(S, \otimes)$ is a commutative quasigroup of order $4n + 2$ with holes $H = \{\{(1, i), (2, i)\} : i \in \mathbb{Q}\}$.

Proof. The fact that S is a commutative quasigroup is immediate as both $\{1, 2\}$ and Q are commutative quasigroups. Then note that $(1, i) \otimes (1, i) = (1 \circ_1 1, i)$, $(1, i) \otimes (2, i) = (1 \circ_1 2, i)$, $(2, i) \otimes (1, i) = (2 \circ_1, i) = (1 \circ_1 2, i)$, and $(2, i) \otimes (2, i) = (2 \circ_1 2, i)$, which forms a subquasigroup since $(\{1, 2\}, \circ_1)$ is a quasigroup.

Maybe I want to show that if Q is a quasigroup and with holes H of size 2 and $h, h' \in H$ with $h \cap h' \neq \varnothing$, then $h = h'$. Then $(h, \circ)$ is a quasigroup and $(h', \circ)$ is a quasigroup. Then is $(h \cap h', \circ)$ a quasigroup? Not necessarily. Well, let $a \in h \cap h'$, so $h = \{a, b\}$ and $h' = \{a, b'\}$. Then

$$\begin{pmatrix} \circ & a & b \\ a & a \circ a & a \circ b \\ b & b \circ a & b \circ b \end{pmatrix}$$

is a quasigroup and

$$\begin{pmatrix} \circ & a & b' \\ a & a \circ a & a \circ b' \\ b' & b' \circ a & b' \circ b' \end{pmatrix}$$

is a quasigroup. In order for both of these to be true, $a \circ a = a$. Thus $a \circ b = b \circ a = b$ and $a \circ b' = b' \circ a = b'$ and $b \circ b = b' \circ b' = a$. So we have

$$\begin{pmatrix} \circ & a & b \\ a & a & b \\ b & b & a \end{pmatrix}$$

and

$$\begin{pmatrix} \circ & a & b' \\ a & a & b' \\ b' & b' & a \end{pmatrix}.$$

Then

$$\begin{pmatrix} \circ & a & b & b' \\ a & a & b & b' \\ b & b & a & * \\ b' & b' & * & a \end{pmatrix}$$

Actually my claim is not true! For example, in

$$\begin{pmatrix} \circ & 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & 3 & 4 \\ 2 & 2 & 1 & 4 & 3 \\ 3 & 3 & 4 & 1 & 2 \\ 4 & 4 & 3 & 2 & 1 \end{pmatrix},$$

the number 1 forms a subquasigroup with 2, 3, and 4, so the holes are $\{1,2\}$, $\{1,3\}$, and $\{1,4\}$.

It could be the definition of holes requires a pairwise partition of the quasigroup.

Whatever... QED or something. □

BASICALLY Exercise 1.5.10 is all about taking an idempotent commutative $2n+1$ quasigroup and basically duplicating every entry as its own individual 2×2 quasigroup, like so

$$Q = \begin{pmatrix} \circ & A & B & C \\ A & A & C & B \\ B & C & B & A \\ C & B & A & C \end{pmatrix}, A = \begin{pmatrix} \circ & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}, B = \begin{pmatrix} \circ & 3 & 4 \\ 3 & 3 & 4 \\ 4 & 4 & 3 \end{pmatrix}, C = \begin{pmatrix} \circ & 5 & 6 \\ 5 & 5 & 6 \\ 6 & 6 & 5 \end{pmatrix}$$

and so

$$Q = \begin{pmatrix} \circ & 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & \mathbf{1} & \mathbf{2} & 5 & 6 & 3 & 4 \\ 2 & \mathbf{2} & \mathbf{1} & 6 & 5 & 4 & 3 \\ 3 & 5 & 6 & \mathbf{3} & \mathbf{4} & 1 & 2 \\ 4 & 6 & 5 & \mathbf{4} & \mathbf{3} & 2 & 1 \\ 5 & 3 & 4 & 1 & 2 & \mathbf{5} & \mathbf{6} \\ 6 & 4 & 3 & 2 & 1 & \mathbf{6} & \mathbf{5} \end{pmatrix}.$$

That's why it only works if $\#Q = 2 \mod 4$. If $\#Q = 0 \mod 4$ I suppose we can start with a half-idempotent commutative quasigroup and doing the same sorta thing?

Let's try it:

$$Q = \begin{pmatrix} A & D & B & C \\ D & B & C & A \\ B & C & A & D \\ C & A & D & B \end{pmatrix}$$

is a half-idempotent commutative quasigroup. Then let $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ and I can already tell this isn't going to work...

Let $(\{1,\dots,2n\},\circ)$ be a commutative quasigroup with holes H. Then

1. $(\{\infty\}\cup(\{1,2,\dots,2n\}\times\{a,b,c\}),B)$ is an STS$(6n+1)$ where B is:
   (a) for $1\le i\le n$, let B_i contain the triples in an STS(7) on the symbols $\{\infty\}\cup(\{2i-1,2i\}\times\{a,b,c\})$ and $B_i\subseteq B$, and
   (b) for $1\le i\ne j\le 2n$, $\{i,j\}\notin H$ place the triples $\{(i,a),(j,a),(i\circ j,b)\}$, $\{(i,b),(j,b),(i\circ j,c)\}$, and $\{(i,c),(j,c),(i\circ j,a)\}$ in B.
2. $(\{\infty_1,\infty_2,\infty_3\}\cup(\{1,2,\dots,2n\}\times\{a,b,c\}),B')$ is an STS$(6n+3)$ where B' replaces (a) in 1. with:
   (a) for $1\le i\le n$ let B' contain the triples in an STS(9) on the symbols $\{\infty_1,\infty_2,\infty_3\}\cup(\{2i-1,2i\}\times\{a,b,c\})$ in which $\{\infty_1,\infty_2,\infty_3\}$ is a triple.
3. $(\{\infty_1,\infty_2,\infty_3,\infty_4,\infty_5\}\cup(\{1,2,\dots,2n\}\times\{a,b,c\}),B'')$ is a PBD$(6n+5)$ with one block of size 5 and the rest of size 3 where B'' replaces (a) of 1. with
   (a) for $1\le i\le n$ B''_i contains the blocks in a PBD(11) on the symbols $\{\infty_1,\dots,\infty_5\}\cup(\{2i-1,2i\}\times\{a,b,c\})$ in which $\{\infty_1,\dots,\infty_5\}$ is a block.

Let's use Exercise 1.1.4 to prove that constructions 1 and 2 are indeed Steiner triple systems.

Proof. First let us start with construction 1. First we want to show that each pair is in at least one triple. First take $\{\infty,(2i,a)\}$ as the pair. Then this pair is in the triple in the Steiner triple system STS(7) on the symbols $\{\infty\}\cup(\{2i-1,2i\}\times\{a,b,c\})$. Same for $2i-1$ and b and c.

Now let us consider the pair $\{(i,a),(j,a)\}$. First, if $\{i,j\}\in H$ then $\infty\cup(\{i,j\}\times\{a,b,c\})$ forms an STS(7) and so $\{(i,a),(j,a)\}$ is in one of those triples. Now if $\{i,j\}$ is not a hole, then this pair is in the triple $\{(i,a),(j,a),(i\circ j,b)\}$. (Same for b and c.) Now let us consider the pair $\{(i,a),(j,b)\}$ where $\{i,j\}$ is not a hole. Then there is a $k\in Q$ such that $i\circ k=j$ and so $\{(i,a),(k,a),(j,b)\}$ is a triple containing the pair. (Same for a,b,c whatever.) Thus every pair is in at least one triple presto!

Now we must show that the number of triples is at most $(6n+1)(6n)/6=6n^2+n$. There are $7n$ triples of type (a) and $3(\frac{2n(2n-2)}{2})=3(n)(2n-2)=6n^2-6n$ triples of type (b). NOTE: it is not $3\binom{2n}{2}$ because the type (b) triples preclude $\{i,j\}\in H$, so it is $3(\frac{2n(2n-2)}{2})$ instead of $3(\frac{2n(2n-1)}{2})$.

Adding these together, we have $7n+6n^2-6n=6n^2+n$ as desired. Thus Exercise 1.1.4 works to form a Steiner triple system here.

Now consider the type 2. system. First, every pair is in at least one triple. Let us start with the pair $\{\infty,(i,a)\}$. Then $\{\infty\}\cup(\{i,j\}\times\{a,b,c\})$ with $\{i,j\}\in H$ is an STS(9) and so the pair $\{\infty,(i,a)\}$ is in one of the triples there. Same for b and c. Then consider $\{(i,a),(j,a)\}$. If $\{i,j\}\in H$, then we have the same STS(9) as before. If $\{i,j\}\notin H$, then the pair $\{(i,a),(j,a)\}$ is in a triple $\{(i,a),(j,a),(i\circ j,b)\}$ of type (b) like in construction 1. Same for $\{(i,a),(j,b)\}$.

Now the fun part. We want to show there are at most $(6n+3)(6n+2)/6 = (36n^2 + 30n+6)/6 = 6n^2+5n+1$ triples. There are 12 triples in each STS(9), giving ostensibly $12n$ triples of type (a'). But $\{\infty_1, \infty_2, \infty_3\}$ is a triple shared by each STS(9). So there are really $11n+1$ triples of type (a'). There are also $3\frac{(2n)(2n-2)}{2} = 6n^2-6n$ triples of type (b). So adding these together, we have $11n+1+6n^2-6n = 6n^2+5n+1$ triples. So Exercise 1.1.4 works. □

There are (up to isomorphism) two Steiner triple systems of size 13. One via the Skolem construction and one via the Wilson construction. Both of them have 26 triples.

Example Wilson.

$$S = \{1,2,3,4,5,6,7,8,9,10,0,\infty_1,\infty_2\}$$

and there are 10 triples of type 1:

$$(1,2,8),(1,3,7),(1,4,6),(2,3,6),(2,4,5),$$
$$(3,9,10),(4,8,10),(5,7,10),(5,8,9),(6,7,9),$$

because they add up to $0 \mod 13-1 = 11$. Then there are 5 type 1,2,3 triples and one type 4:

$$\begin{aligned}(0,1,10),(0,3,5),(0,6,8),(0,2,9),(0,4,7)\\(\infty_1,1,9),(\infty_1,2,10),(\infty_1,7,8),(\infty_1,3,4),(\infty_1,5,6)\\(\infty_2,1,5),(\infty_2,2,7),(\infty_2,3,8),(\infty_2,4,9),(\infty_2,6,10)\\(0,\infty_1,\infty_2)\end{aligned}$$

where the triples of type 1,2,3 come from the three colors of a 3-coloring of the graph whose vertices are units of $\mathbb{Z}/11\mathbb{Z}$ connected to each other if they are either negatives of each other or one of a multiple of -2 by the other.

Example Skolem. See example JK3.