1 Blowing Up $\mathbb{P}^3_K$ at Six Lines

The standard cubo-cubic transformation is given by blowing up a curve of arithmetic genus 3 in $\mathbb{P}^2$. Usually, this is taken to be a smooth curve, but we can also work with six lines in $\mathbb{P}^3$ arranged in either a tetrahedral formation or (as we shall investigate) four general lines and their two transversals.

Let us blow up $\mathbb{P}^3$ at four general lines $L_1, \ldots, L_4$ and the two transversal lines T_1, T_2. Call this blowup X. Then

$$\mathrm{Pic}(X) = \langle H, L_1, L_2, L_3, L_4, T_1, T_2 \rangle$$

provides a basis in which we can express the cubo-cubic transformation Γ on X. Namely, we have

$$\Gamma = \begin{pmatrix} 3 & 2 & 2 & 2 & 2 & 0 & 0 \\ -1 & 0 & -1 & -1 & -1 & 0 & 0 \\ -1 & -1 & 0 & -1 & -1 & 0 & 0 \\ -1 & -1 & -1 & 0 & -1 & 0 & 0 \\ -1 & -1 & -1 & -1 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 & 1 & 0 \\ -1 & -1 & -1 & -1 & -1 & 0 & 1 \end{pmatrix}.$$

To note some particular examples of the behavior of Γ, observe that the class of a general plane H is transformed to a cubic surface containing the four skew lines (and thus necessarily contains the two transversals). The class of an exceptional locus L_i is transformed to a quadric surface containing the other three lines (and thus necessarily the two transversals as well). Each transversal is fixed by Γ. Of special use to us will be the transformation

$$\Gamma(H - L_i) = H - L_i \tag{1}$$

for each L_i. This identity will be relevant in constructing explicit formulae for Γ in the next section. For more information, refer to Unexpected surfaces singular on lines in $\mathbb{P}^3$ by Dumnicki, Harbourne, Roé, Szemberg, and Tutaj-Gasińska.

2 Explicit Construction

2.1 General Lines

Let $L_1, \ldots, L_4$ be four general lines in $\mathbb{P}^3_K$, where K is algebraically closed. We shall construct the cubo-cubic transformation induced by these four lines.

Naïvely, the cubo-cubic transformation is simply the rational map induced by the linear system $\mathfrak{d} = |3H - L_1 - \cdots - L_4|$ (note that we have unassigned base points as every cubic containing the four lines must also contain the two transversal lines thereof). However, finding an appropriate ordered basis of $\mathcal{L}(\mathfrak{d})$ is imperative, as this is how we guarantee that our transformation is an involution.

To begin, let

$$\mathcal{L}(H - L_i) = \langle p_i, p_i' \rangle$$

be polynomial generators for the linear system of planes surfaces containing L_i, and let

$$\mathcal{L}(2H - L_1 - L_2 - L_3 - L_4 + L_i) = \langle q_i \rangle$$

be the (unique) quadric surface containing all lines except L_i. Then we may start with the basis

$$\mathcal{L}(\mathfrak{d}) = \langle p_1q_1, p_1'q_1, p_2q_2, p_2'q_2 \rangle \cong K^4.$$

An important guiding feature of the cubo-cubic transformation Γ is that we want the transformations of p_i and p_i' to be some linear combination of p_iq_i and $p_i'q_i$, which matches the action of the cubo-cubic matrix on the Picard group (see Identity (1) of the previous section). That is, we want

$$\Gamma(p_i), \Gamma(p_i') \in \langle p_iq_i, p_i'q_i \rangle.$$

To this end, we want to find a linear transformation from $\langle p_1, p_1', p_2, p_2' \rangle = \mathcal{L}(H) \cong K^4$ to $\langle p_1q_1, p_1'q_1, p_2q_2, p_2'q_2 = \mathcal{L}(\mathfrak{d}) \cong K^4$ that transforms the following four planes in the prescribed manner:

- $\langle p_1, p_1' \rangle \mapsto \langle p_1q_1, p_1'q_1 \rangle$,
- $\langle p_2, p_2' \rangle \mapsto \langle p_2q_2, p_2'q_2 \rangle$,
- $\langle p_3, p_3' \rangle = \langle A_3p_1 + A_3'p_1' + B_3p_2 + B_3'p_2', C_3p_1 + C_3'p_1' + D_3p_2 + D_3'p_2' \rangle \mapsto \langle p_3q_3, p_3'q_3 \rangle = \langle E_3p_1q_1 + E_3'p_1'q_1 + F_3p_2q_2 + F_3'p_2'q_2, G_3p_1q_1 + G_3'p_1'q_1 + H_3p_2q_2 + H_3'p_2'q_2 \rangle$,
- $\langle p_4, p_4' \rangle = \langle A_4p_1 + A_4'p_1' + B_4p_2 + B_4'p_2', C_4p_1 + C_4'p_1' + D_4p_2 + D_4'p_2' \rangle \mapsto \langle p_4q_4, p_4'q_4 \rangle = \langle E_4p_1q_1 + E_4'p_1'q_1 + F_4p_2q_2 + F_4'p_2'q_2, G_4p_1q_1 + G_4'p_1'q_1 + H_4p_2q_2 + H_4'p_2'q_2 \rangle$.

In terms of vectors, our desired linear transformation exhibits the following properties:

- $\left\langle \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \right\rangle \mapsto \left\langle \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \right\rangle$,
- $\left\langle \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix} \right\rangle \mapsto \left\langle \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix} \right\rangle$,
- $\left\langle \begin{pmatrix}A_3\\A_3'\\B_3\\B_3'\end{pmatrix}, \begin{pmatrix}C_3\\C_3'\\D_3\\D_3'\end{pmatrix} \right\rangle \mapsto \left\langle \begin{pmatrix}E_3\\E_3'\\F_3\\F_3'\end{pmatrix}, \begin{pmatrix}G_3\\G_3'\\H_3\\H_3'\end{pmatrix} \right\rangle$,
- $\left\langle \begin{pmatrix}A_4\\A_4'\\B_4\\B_4'\end{pmatrix}, \begin{pmatrix}C_4\\C_4'\\D_4\\D_4'\end{pmatrix} \right\rangle \mapsto \left\langle \begin{pmatrix}E_4\\E_4'\\F_4\\F_4'\end{pmatrix}, \begin{pmatrix}G_4\\G_4'\\H_4\\H_4'\end{pmatrix} \right\rangle$.

2.2 Normal Form

Let us now arrange the four lines in what we will call normal form. We have the following four lines.

- $L_\infty = \text{rowspan}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$,
- $L_0 = \text{rowspan}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$,
- $L_1 = \text{rowspan}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$,
- $L_2 = \text{rowspan}\begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{pmatrix}$,

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a-1 & b \\ c & d-1 \end{pmatrix} \in \mathrm{GL}(2, K)$. Every set of four mutually-skew lines can be assumed to exhibit this form. Then $\mathcal{L}(H - L_\infty) = \langle x, y \rangle$ and $\mathcal{L}(H - L_0) = \langle z, w \rangle$, so we will use $\langle x, y, z, w \rangle$ as our basis for $\mathcal{L}(H)$. Also note that $\mathcal{L}(H - L_1) = \langle x - z, y - w \rangle$ and $\mathcal{L}(H - L_2) = \langle (ad - bc)x - dz + cw, (ad - bc)y + bz - aw \rangle$.

Furthermore,

$$\mathcal{L}(2H - L_0 - L_1 - L_2) = \langle bxz + (bc - ad + d)yz - bz^2 + (-bc + ad - a)xw - cyw + (a - d)zw + cw^2 \rangle$$

and

$$\mathcal{L}(2H - L_\infty - L_1 - L_2) = \langle bx^2 + (-a + d)xy - cy^2 - bxz + (-d + 1)yz + (a - 1)xw + cyw \rangle,$$

which we will denote q_∞ and q_0, respectively. Then we have

$$\mathcal{L}(\mathfrak{d}) = \langle xq_\infty, yq_\infty, zq_0, wq_0 \rangle$$

as our basis for cubics containing the four lines. Furthermore, note

$$\mathcal{L}(2H - L_\infty - L_0 - L_2) = \langle bxz + dyz - axw - cyw = q_1 \rangle$$

and

$$\mathcal{L}(2H - L_\infty - L_0 - L_1) = \langle xw - yz = q_2 \rangle.$$

Now we must write $(x - z)q_1$, $(y - w)q_1$, $((ad - bc)x - dz + cw)q_2$, and $((ad - bc)y + bz - aw)q_2$

as linear combinations of xq_∞, yq_∞, zq_0, and wq_0. We can write each as follows:

$$(x-z)q_1 = \frac{\begin{pmatrix} a-1 & c & bc-ad+d & -c \end{pmatrix}}{bc-ad+a+d-1} \begin{pmatrix} xq_\infty \\ yq_\infty \\ zq_0 \\ wq_0 \end{pmatrix},$$

$$(y-w)q_1 = \frac{\begin{pmatrix} b & d-1 & -b & bc-ad+a \end{pmatrix}}{bc-ad+a+d-1} \begin{pmatrix} xq_\infty \\ yq_\infty \\ zq_0 \\ wq_0 \end{pmatrix},$$

$$((ad-bc)x-dz+cw)q_2 = \frac{\begin{pmatrix} bc-ad+d & -c & -bc+ad-d & c \end{pmatrix}}{bc-ad+a+d-1} \begin{pmatrix} xq_\infty \\ yq_\infty \\ zq_0 \\ wq_0 \end{pmatrix},$$

$$((ad-bc)y+bz-aw)q_2 = \frac{\begin{pmatrix} -b & bc-ad+a & b & -bc+ad-a \end{pmatrix}}{bc-ad+a+d-1} \begin{pmatrix} xq_\infty \\ yq_\infty \\ zq_0 \\ wq_0 \end{pmatrix}.$$

Clearing out the denominator, our desired transformation of K^4 exhibits the following action on these four planar subspaces:

- $\left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle \mapsto \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle$,
- $\left\langle \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle \mapsto \left\langle \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle$,
- $\left\langle \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \right\rangle \mapsto \left\langle \begin{pmatrix} a-1 \\ c \\ bc-ad+d \\ -c \end{pmatrix}, \begin{pmatrix} b \\ d-1 \\ -b \\ bc-ad+a \end{pmatrix} \right\rangle$,
- $\left\langle \begin{pmatrix} ad-bc \\ 0 \\ -d \\ c \end{pmatrix}, \begin{pmatrix} 0 \\ ad-bc \\ b \\ -a \end{pmatrix} \right\rangle \mapsto \left\langle \begin{pmatrix} bc-ad+d \\ -c \\ -bc+ad-d \\ c \end{pmatrix}, \begin{pmatrix} -b \\ bc-ad+a \\ b \\ -bc+ad-a \end{pmatrix} \right\rangle$.

To find such a linear transformation on K^4, let us perform the following change of basis. Let

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} ad-bc \\ 0 \\ -d \\ c \end{pmatrix}, \begin{pmatrix} 0 \\ ad-bc \\ b \\ -a \end{pmatrix} \right\}.$$

Then under this new basis, we have the vectors

$$\begin{pmatrix} a-1 \\ c \\ bc-ad+d \\ -c \end{pmatrix}, \begin{pmatrix} b \\ d-1 \\ -b \\ bc-ad+a \end{pmatrix}, \begin{pmatrix} bc-ad+d \\ -c \\ -bc+ad-d \\ c \end{pmatrix}, \begin{pmatrix} -b \\ bc-ad+a \\ b \\ -bc+ad-a \end{pmatrix}$$

become

$$\begin{pmatrix} -bc+ad-1 \\ 0 \\ \frac{-bc+ad-1}{bc-ad} \\ \frac{-c}{bc-ad} \end{pmatrix}, \begin{pmatrix} 0 \\ -bc+ad-1 \\ \frac{-b}{bc-ad} \\ \frac{-bc+ad-d}{bc-ad} \end{pmatrix}, \begin{pmatrix} bc-ad+d \\ -c \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -b \\ bc-ad+a \\ 0 \\ 0 \end{pmatrix},$$

respectively. Once again, we may clear our those pesky denominators and construct the following matrix. For space, we shall denote the determinant $ad-bc$ as Δ.

$$M = \begin{pmatrix} (\Delta^2+1)\lambda_1 & (\Delta^2+1)\lambda_3 & (-\Delta+d)\lambda_5 - b\lambda_6 & (-\Delta+d)\lambda_7 - b\lambda_8 \\ (\Delta^2+1)\lambda_2 & (\Delta^2+1)\lambda_4 & -c\lambda_5 + (-\Delta+a)\lambda_6 & -c\lambda_7 + (-\Delta+a)\lambda_8 \\ (-\Delta-1)\lambda_1 - b\lambda_2 & (-\Delta-1)\lambda_3 - b\lambda_4 & 0 & 0 \\ -c\lambda_1 + (-\Delta-d)\lambda_2 & -c\lambda_3 + (-\Delta-d)\lambda_4 & 0 & 0 \end{pmatrix}.$$

We want to find which values of λ_i cause the product

$$[I_4]_{\mathcal{E}\leftarrow\mathcal{B}} M [I_4]_{\mathcal{B}\leftarrow\mathcal{E}}$$

to be a block diagonal matrix of two 2×2 matrices. Solving for $\vec{\lambda}$, we end up finding $\vec{\lambda}$ is in the kernel of a rank 6 matrix, whose kernel is

$$\Lambda = \left\langle \begin{pmatrix} \frac{c}{bc-ad} \\ \frac{-ac}{b^2c-abd} \\ \frac{d}{bc-ad} \\ \frac{-c}{bc-ad} \\ 0 \\ \frac{-c}{b} \\ 1 \\ 0 \end{pmatrix} = \psi_1, \begin{pmatrix} \frac{-a}{bc-ad} \\ \frac{a^2+bc-ad}{b^2c-abd} \\ \frac{-b}{bc-ad} \\ \frac{a}{bc-ad} \\ -1 \\ \frac{a-d}{b} \\ 0 \\ 1 \end{pmatrix} = \psi_2 \right\rangle.$$

Choosing $\psi = \mu_1\psi_1 + \mu_2\psi_2 \in \Lambda$, we get the block diagonal matrix

$$[I_4]_{\mathcal{E}\leftarrow\mathcal{B}} M [I_4]_{\mathcal{B}\leftarrow\mathcal{E}} = \begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{pmatrix},$$

where

$$\mathbf{A} = \begin{pmatrix} \frac{-c}{bc-ad}\mu_1 + \frac{bc-ad+a}{bc-ad}\mu_2 & \frac{-bc+ad-d}{bc-ad}\mu_1 + \frac{b}{bc-ad}\mu_2 \\ \frac{bc^2-acd+ac}{b^2c-abd}\mu_1 + \frac{-abc+a^2d+bcd-ad^2-a^2-bc+ad}{b^2c-abd}\mu_2 & \frac{c}{bc-ad}\mu_1 + \frac{-bc+ad-a}{bc-ad}\mu_2 \end{pmatrix}$$

and

$$\mathbf{B} = \begin{pmatrix} \frac{-bc^2+acd-ac-cd}{bc-ad}\mu_1 + \frac{abc-a^2d+a^2+bc}{bc-ad}\mu_2 & \frac{-bcd+ad^2-bc-d^2}{bc-ad}\mu_1 + \frac{b^2c-abd+ab+bd}{bc-ad}\mu_2 \\ \frac{abc^2-a^2cd+a^2c+bc^2}{b^2c-abd}\mu_1 + \frac{-a^2bc-b^2c^2+a^3d+2abcd-a^2d^2-a^3-2abc+a^2d}{b^2c-abd}\mu_2 & \frac{bc^2-acd+ac+cd}{bc-ad}\mu_1 + \frac{-abc+a^2d-a^2-bc}{bc-ad}\mu_2 \end{pmatrix}.$$

Since any choice of μ_1 and μ_2 is sufficient, we may be kind to ourselves by choosing nice coefficients; by choosing $\mu_1 = ab$ and $\mu_2 = bc$, we get a nice and simple matrix

$$[I_4]_{\mathcal{E}\leftarrow\mathcal{B}} M [I_4]_{\mathcal{B}\leftarrow\mathcal{E}} = \begin{pmatrix} bc & -ab+b & 0 & 0 \\ cd-c & -bc & 0 & 0 \\ 0 & 0 & bc & b^2c-abd+bd \\ 0 & 0 & -bc^2+acd-ac & -bc \end{pmatrix}$$

From this matrix we define the following four cubic polynomials.

- $\gamma_0 = bc(xq_\infty) + (cd-c)(yq_\infty)$,
- $\gamma_1 = (-ab+b)(xq_\infty) - bc(yq_\infty)$,
- $\gamma_2 = bc(zq_0) + (-bc^2+acd-ac)(wq_0)$,
- $\gamma_3 = (b^2c-abd+bd)(zq_0) - bc(wq_0)$.

Then $\langle \gamma_0, \gamma_1, \gamma_2, \gamma_3 \rangle = \mathcal{L}(3H - L_\infty - L_0 - L_1 - L_2)$ is an ordered basis that we may use to perform the cubo-cubic transformation

$$\Gamma(x:y:z:w) = (\gamma_0(x,y,z,w) : \gamma_1(x,y,z,w) : \gamma_2(x,y,z,w) : \gamma_3(x,y,z,w)).$$

Note: choosing $(\mu_1 : \mu_2) = (a - d \pm \sqrt{(a-d)^2 + 4bc} : 2c)$ seems to force $[I_4]_{\mathcal{E}\leftarrow\mathcal{B}} M [I_4]_{\mathcal{B}\leftarrow\mathcal{E}}$ to be an involution (in $\mathrm{PGL}(4,K)$). However, this does not appear to be necessary for Γ to be an involution.

Remark 1. Fix the lines L_∞, L_0, and $L_1 = \begin{pmatrix} I_2 & I_2 \end{pmatrix}$ and consider the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, inducing the line $L_A = \begin{pmatrix} I_2 & A \end{pmatrix}$ such that $L_A \cap L_0 = L_A \cap L_1 = \varnothing$. Then the four lines $\{L_\infty, L_0, L_1, L_A\}$ induce a unique transversal if and only if A is not diagonalizable. Furthermore, the unique transversal line is given by $\begin{pmatrix} a-d & 2b & 0 & 0 \\ 0 & 0 & a-d & 2b \end{pmatrix}$, or $\begin{pmatrix} \mathbf{v} & 0 & 0 \\ 0 & 0 & \mathbf{v} \end{pmatrix}$, where $\mathbf{v}$ is a left-eigenvector of A.

3 Examples

Example 1. Let us choose the four skew lines in normal form given by $\{L_\infty, L_0, L_1, L_2\}$, where

$$L_2 = \text{rowspan}\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 1 \end{pmatrix}.$$

Then the group of the groupoid is $C_3 \cong G = \left\langle \left[\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}\right] \right\rangle \leq \text{PGL}(2, K)$. We have

$$q_\infty = -xz + z^2 + xw - yw - zw + w^2,$$

and

$$q_0 = -x^2 + xy - y^2 + xz - xw + yw,$$

and so our four generators of the cubo-cubic transform centered at our four lines are

$$\begin{aligned}
\gamma_0 &= -xq_\infty = x^2z - xz^2 - x^2w + xyw + xzw - xw^2 \\
\gamma_1 &= (-x+y)q_\infty = x^2z - xyz - xz^2 + yz^2 - x^2w + 2xyw - y^2w + xzw - yzw - xw^2 + yw^2 \\
\gamma_2 &= (-z+w)q_0 = x^2z - xyz + y^2z - xz^2 - x^2w + xyw - y^2w + 2xzw - yzw - xw^2 + yw^2 \\
\gamma_3 &= wq_0 = -x^2w + xyw - y^2w + xzw - xw^2 + yw^2.
\end{aligned}$$

Now let us consider the point $p = (0:0:1:0) \in L_\infty$. The orbit of p, denoted $[p]$, is the set of the following twelve points.

$$\begin{array}{ccc}
(0:0:1:0) & (0:0:0:1) & (0:0:1:-1) \\
(1:0:0:0) & (0:1:0:0) & (1:-1:0:0) \\
(1:0:1:0) & (0:1:0:1) & (1:-1:1:-1) \\
(1:0:0:-1) & (0:1:1:1) & (1:-1:-1:-2)
\end{array}$$

Since the fibers of these points are lines in the exceptional loci of the four skew lines, their cubo-cubic transforms are lines as well, as given by the Picard group. Specifically, we have the images

$$\begin{array}{ccc}
\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} & \begin{pmatrix} 3 & 0 & 2 & 1 \\ 0 & 3 & 2 & 1 \end{pmatrix} \\
\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{pmatrix} \\
\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \end{pmatrix} \\
\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{pmatrix}
\end{array}$$

respectively. Note that, though it may appear as though some of these lines intersect, in the blowup X they do not; all perceived intersection points are on the lines $\mathcal{L}$, and so they are actually separated on the exceptional loci.