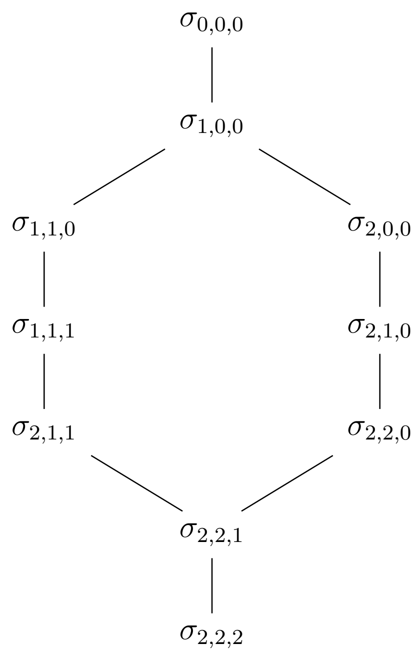
It is interesting to me to consider configurations of planes in $\mathbb{P}^4$. Let us start by considering the Chow ring $A = A(\mathfrak{Gr}(3,5))$.

First note that $A \cong \mathbb{Z}^{10}$, we have



and the following geometric interpretations of each Schubert class:

$\Sigma_{0,0,0}$	All planes
$\Sigma_{1,0,0}$	Planes touching a given line
$\Sigma_{1,1,0}$	Planes whose intersection with a given plane π is at least some line $\ell \subseteq \pi$
$\Sigma_{2,0,0}$	Planes containing a given point
$\Sigma_{1,1,1}$	Planes in a given threeperplane
$\Sigma_{2,1,0}$	Planes intersecting a given plane π at at least a line and contain a given point $p \in \pi$
$\Sigma_{2,1,1}$	Planes contained in a given threeperplane τ and contain a given point $p \in \tau$
$\Sigma_{2,2,0}$	Planes containing a given line
$\Sigma_{2,2,1}$	Planes contained in a given threeperplane τ and contain a given line $\ell \subseteq \tau$
$\Sigma_{2,2,2}$	One plane

We also have the following multiplication table:

$\times$	σ_0	σ_1	$\sigma_{1,1}$	σ_2	$\sigma_{1,1,1}$	$\sigma_{2,1}$	$\sigma_{2,1,1}$	$\sigma_{2,2}$	$\sigma_{2,2,1}$	$\sigma_{2,2,2}$
σ_0	σ_0	σ_1	$\sigma_{1,1}$	σ_2	$\sigma_{1,1,1}$	$\sigma_{2,1}$	$\sigma_{2,1,1}$	$\sigma_{2,2}$	$\sigma_{2,2,1}$	$\sigma_{2,2,2}$
σ_1		$\sigma_{1,1}+\sigma_2$	$\sigma_{1,1,1}+\sigma_{2,1}$	$\sigma_{2,1}$	$\sigma_{2,1,1}$	$\sigma_{2,1,1}+\sigma_{2,2}$	$\sigma_{2,2,1}$	$\sigma_{2,2,1}$	$\sigma_{2,2,2}$	0
$\sigma_{1,1}$			$\sigma_{2,1,1}+\sigma_{2,2}$	$\sigma_{2,1,1}$	$\sigma_{2,2,1}$	$\sigma_{2,2,1}$	$\sigma_{2,2,2}$	0	0	0
σ_2				$\sigma_{2,2}$	0	$\sigma_{2,2,1}$	0	$\sigma_{2,2,2}$	0	0
$\sigma_{1,1,1}$					$\sigma_{2,2,2}$	0	0	0	0	0
$\sigma_{2,1}$						$\sigma_{2,2,2}$	0	0	0	0
$\sigma_{2,1,1}$							0	0	0	0
$\sigma_{2,2}$								0	0	0
$\sigma_{2,2,1}$									0	0
$\sigma_{2,2,2}$										0

According to my (possibly erroneous) calculations, in $\mathbb{P}^4_{\mathbb{F}_2}$ we have the following sizes of Schubert cycles:

$$\begin{aligned}
\#\Sigma_{0,0,0} &= 155\\
\#\Sigma_{1,0,0} &= 91\\
\#\Sigma_{1,1,0} &= 43\\
\#\Sigma_{2,0,0} &= 35\\
\#\Sigma_{1,1,1} &= 15\\
\#\Sigma_{2,1,0} &= 19\\
\#\Sigma_{2,1,1} &= 7\\
\#\Sigma_{2,2,0} &= 7\\
\#\Sigma_{2,2,1} &= 3\\
\#\Sigma_{2,2,2} &= 1
\end{aligned}$$

Question: Can we cover $\mathbb{P}^4_{\mathbb{F}_2}$ with planes that pairwise intersect at points?

Question: Can we come up with a configuration $(31_?, ?_7)$ covering $\mathbb{P}^4_{\mathbb{F}_2}$? Possibly a $(31_7, 31_7)$.

Unfortunately, I think that, given a point $p \in \mathbb{P}^4_{\mathbb{F}_2}$, a maximal set of planes π_i that contain p and intersect each other only at p (that is, $\pi_i \cap \pi_j = \{p\}$) has size 5. Think of it like this: take a threeperplane τ that does not contain p. Then every plane containing p must intersect τ at a line. So we want to find a maximal set of lines in $\tau \cong \mathbb{P}^3_{\mathbb{F}_2}$ that are mutually-skew. That is a spread in $\mathbb{P}^3_{\mathbb{F}_2}$, which is well-known to consist of 5 lines.

Therefore in that $(31_7, 31_7)$ configuration, we must attain planes that meet at lines: every set of seven planes containing a point p must have planes whose intersection is a line.