

Note the second-order differential equation

$$LQ'' + RQ' + C^{-1}Q = V,$$

where

L = inductance

R = resistance

C = capacitance

Q = charge

V = voltage

and Q' is change in charge with respect to time (a.k.a. current) and Q'' is change in current with respect to time.

By setting $V = \sin(2\pi ft)$, where t is time and f is frequency, we attain the same differential equation as that for a dampened spring system (where L is analogous to the mass, R to the dampening, and C^{-1} to the spring constant).

We also get individual equations for how resistors, capacitors, and inductors act on a circuit (we will be using $I := Q'$ for current):

$$RI = V, \text{ (Ohm's Law)}$$

$$LI' = V,$$

$$CV' = I.$$

For an inductor, voltage is proportional to the change in current: when the voltage is zero, there is no change in the current. When the current first starts, the voltage was zero and so the change in current starts off zero and increases slowly— as this happens the voltage across the inductor begins to increase and so the change in current begins to increase, until at some point the resistance offered by the inductor goes down close to zero and so the voltage across the inductor goes to zero, leaving a constant current flow.

For a capacitor, current is proportional to the change in voltage: the voltage can only be constant when there is no current through the capacitor; any current will induce a change in the voltage across the capacitor.

Setting $V(t) = \sin(2\pi ft)$ for some f , we can solve this second-order differential equation for $Q(t)$:

$$\frac{(C - 4C^2Lf^2\pi^2)\sin(2\pi ft) - 2C^2Rf\pi\cos(2\pi ft)}{16C^2L^2f^4\pi^4 - 8CLf^2\pi^2 + 4C^2R^2f^2\pi^2 + 1} + J_1\alpha(t) + J_2\beta(t),$$

where $\alpha(t)$ and $\beta(t)$ vary depending on whether the system is overdamped, underdamped, or critically damped.

- When the system is overdamped ($R^2 < 4LC^{-1}$):

$$\alpha(t) = \exp\left(-\frac{R}{2L}t\right) \sin\left(\frac{\sqrt{|R^2 - 4LC^{-1}|}}{2L}t\right),$$

$$\beta(t) = \exp\left(-\frac{R}{2L}t\right) \cos\left(\frac{\sqrt{|R^2 - 4LC^{-1}|}}{2L}t\right).$$

- When the system is underdamped ($R^2 > 4LC^{-1}$):

$$\alpha(t) = \exp\left(\frac{-R + \sqrt{|R^2 - 4LC^{-1}|}}{2L}t\right),$$

$$\beta(t) = \exp\left(\frac{-R - \sqrt{|R^2 - 4LC^{-1}|}}{2L}t\right).$$

- When the system is critically damped ($R^2 = 4LC^{-1}$):

$$\alpha(t) = \exp\left(\frac{-R}{2L}t\right),$$

$$\beta(t) = t \exp\left(\frac{-R}{2L}t\right).$$

Now let f vary, and define

$$\sigma(f) = \frac{C - 4C^2Lf^2\pi^2}{16C^2L^2f^4\pi^4 - 8CLf^2\pi^2 + 4C^2R^2f^2\pi^2 + 1}$$

and

$$\kappa(f) = \frac{-2C^2Rf\pi}{16C^2L^2f^4\pi^4 - 8CLf^2\pi^2 + 4C^2R^2f^2\pi^2 + 1}.$$

Then we can create a function that represents the amplitude of $Q(t)$ in terms of f , that is, a function that stores the sensitivity of the circuit to different frequencies:

$$S(f) = \sqrt{\sigma(f)^2 + \kappa(f)^2}.$$

The circuit will have the highest sensitivity to the frequency where $S(f)$ peaks; that is, where

$$S'(f) = \frac{\sigma(f)\sigma'(f) + \kappa(f)\kappa'(f)}{\sqrt{\sigma(f)^2 + \kappa(f)^2}} = 0.$$

When $R = 0$, $S(f)$ has a vertical asymptote that the resonant frequency, which can be calculated as

$$F = \frac{1}{2\pi\sqrt{CL}}.$$

We can solve this differential equation without resistance and with resistance, also finding frequency sensitivity. We can also see general solutions to spring/mass differential equations.