Consider the Grassmannian $\mathfrak{Gr}(k,n)$ of k-planes in a vector space V of dimension n and a full flag $\mathcal{V}$: $0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n = V$, where $\dim V_i = i$. Schubert cycles are dictated by a decreasing sequence $a = (a_1, \ldots, a_k)$ by $n-k \geq a_1 \geq a_2 \geq \cdots \geq a_k \geq 0$. Then

$$\Sigma_a(\mathcal{V}) = \{\Lambda \in \mathfrak{Gr}(k,n) : \dim(V_{n-k+i-a_i} \cap \Lambda) \geq i \text{ for all } i\}.$$

For example, in $\mathfrak{Gr}(2,5)$ there are 10 decreasing sequences of length 2, corresponding to 10 Schubert cycles.

$$\Sigma_{3,3} = \{\Lambda : \dim(V_1 \cap \Lambda) \geq 1, \dim(V_2 \cap \Lambda) \geq 2\} = \{V_2\}.$$

$$\Sigma_{3,2} = \{\Lambda : \dim(V_1 \cap \Lambda) \geq 1, \dim(V_3 \cap \Lambda \geq 1)\} = \{\Lambda : V_1 \subseteq \Lambda \subseteq V_3\}.$$

$$\Sigma_{3,1} = \{\Lambda : V_1 \subseteq \Lambda \subseteq V_4\}.$$

$$\Sigma_{3,0} = \{\Lambda : V_1 \subseteq \Lambda\}.$$

$$\Sigma_{2,2} = \{\Lambda : \dim(V_2 \cap \Lambda) \geq 1, \dim(V_3 \cap \Lambda) \geq 2\} = \{\Lambda : \dim(V_2 \cap \Lambda) \geq 1, \Lambda \subseteq V_3\}.$$

$$\Sigma_{2,1} = \{\Lambda : \dim(V_2 \cap \Lambda) \geq 1, \Lambda \subseteq V_4\}.$$

$$\Sigma_{2,0} = \{\Lambda : \dim(V_2 \cap \Lambda) \geq 1\}.$$

$$\Sigma_{1,1} = \{\Lambda : \dim(V_3 \cap \Lambda) \geq 1, \Lambda \subseteq V_4\} = \{\Lambda : \Lambda \subseteq V_4\}^*.$$

$$\Sigma_{1,0} = \{\Lambda : \dim(V_3 \cap \Lambda) \geq 1\}.$$

$$\Sigma_{0,0} = \{\Lambda : \dim(V_4 \cap \Lambda) \geq 1\} = \mathfrak{Gr}(2,5).$$

*Note if $\Lambda \subseteq V_4$ then Λ necessarily has nontrivial intersection with V_3, because $\dim(\Lambda) = 2$. We can also write Σ_a in matrix form where a determined the number of extra zeroes in each row of the 2×5 matrix

$$\begin{pmatrix} * & * & * & * & 0 \\ * & * & * & * & * \end{pmatrix}$$

where we can choose $e_1, \ldots, e_i$ as our basis elements of V_i. Then

$$\Sigma_{3,3} = \begin{pmatrix} * & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \end{pmatrix}.$$

$$\Sigma_{3,2} = \begin{pmatrix} * & 0 & 0 & 0 & 0 \\ * & * & * & 0 & 0 \end{pmatrix}.$$

$$\Sigma_{3,1} = \begin{pmatrix} * & 0 & 0 & 0 & 0 \\ * & * & * & * & 0 \end{pmatrix}.$$

$$\Sigma_{3,0} = \begin{pmatrix} * & 0 & 0 & 0 & 0 \\ * & * & * & * & * \end{pmatrix}.$$

$$\Sigma_{2,2} = \begin{pmatrix} * & * & 0 & 0 & 0 \\ * & * & * & 0 & 0 \end{pmatrix}.$$

$$\Sigma_{2,1} = \begin{pmatrix} * & * & 0 & 0 & 0 \\ * & * & * & * & 0 \end{pmatrix}.$$

$$\Sigma_{2,0} = \begin{pmatrix} * & * & 0 & 0 & 0 \\ * & * & * & * & * \end{pmatrix}.$$

$$\Sigma_{1,1} = \begin{pmatrix} * & * & * & 0 & 0 \\ * & * & * & * & 0 \end{pmatrix}.$$

$$\Sigma_{1,0} = \begin{pmatrix} * & * & * & 0 & 0 \\ * & * & * & * & * \end{pmatrix}.$$

$$\Sigma_{0,0} = \begin{pmatrix} * & * & * & * & 0 \\ * & * & * & * & * \end{pmatrix}.$$

The only thing to keep in mind is the two rows must be linearly independent in order to make a plane in the first place. For example, the plane $\langle e_1 + 2e_2, e_3 - e_4\rangle \in \Sigma_{2,1}$, represented by $\begin{pmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{pmatrix}$.

Given to Schubert sequences a and b, call $b \le a$ if $b_i \le a_i$ for some i. Furthermore, define $|a| = \sum_{i=1}^{k} a_i$. Then $\Sigma_b \subseteq \Sigma_a$ for all $b \ge a$. For an integer λ, we will denote $\Sigma_{\lambda,0,\ldots,0}$ by Σ_λ and $\Sigma_{\lambda,\lambda,\ldots,\lambda}$ by Σ_{λ^k}.

The *Schubert cell* Σ_a° is defined by $\Sigma_a \setminus \left(\bigcup_{b>a} \Sigma_b\right)$. The Schubert cell is an affine space isomorphic to $\mathbb{A}^{k(n-k)-|a|}$. We can show this with the specific example

$$\Sigma_{3,2,2,1} = \begin{pmatrix} * & * & * & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & 0 \end{pmatrix} \in \mathfrak{Gr}(4,9).$$

In this case the Schubert cell is given by

$$\Sigma_{3,2,2,1}^\circ = \begin{pmatrix} * & * & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & 1 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & 1 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & 1 & 0 \end{pmatrix}$$

and we can then force the rows to be linearly independent by performing row subtractions and getting

$$\Sigma_{3,2,2,1} \cong \begin{pmatrix} * & * & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & * & 1 & 0 & 0 & 0 & 0 \\ * & * & 0 & * & 0 & 1 & 0 & 0 & 0 \\ * & * & 0 & * & 0 & 0 & * & 1 & 0 \end{pmatrix}$$

and we can see that $\dim \Sigma^\circ_{3,2,2,1} = 12 = 4(9-4) - (3+2+2+1)$. So indeed $\Sigma^\circ_{3,2,2,1} = \mathbb{A}^{12}$. In this example, we can see for example that $\langle e_1 - e_2 + e_3, 2e_1 + e_4 + e_5, e_6, e_8 \rangle \in \Sigma^\circ_{3,2,2,1}$, corresponding to the matrix

$$\begin{pmatrix} 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

So in general we can see why $\Sigma^\circ_a \cong \mathbb{A}^{k(n-k)-|a|}$ and in particular $\Sigma^\circ_0 \cong \mathbb{A}^{k(n-k)}$ and so $\dim(\Sigma_0 = \mathfrak{Gr}(k,n)) = k(k-n)$. (Not $\binom{n}{k}$, which is the dimension of the wedge product $\bigwedge^k V$, of which the Grassmannian is a proper subset.) There are in total $\binom{n}{k}$ Schubert cycles for $\mathfrak{Gr}(k,n)$, and since the Schubert cycles generate $A(\mathfrak{Gr}(k,n))$ as an abelian group, we have $A(\mathfrak{Gr}(k,n)) \cong \mathbb{Z}^{\binom{n}{k}}$.

We can also see that $(\sigma_{n-k})^k = (\sigma_{1^k})^{n-k} = \sigma_{(n-k)^k} \in A^{k(n-k)}(\mathfrak{Gr}(k,n))$. We can use the fact that σ_{n-k} is the class of all k-planes containing a line of a flag. Then σ^k_{n-k} is the class of all k-planes containing k lines, which is a unique k-plane, so $\sigma_{(n-k)^k}$. Furthermore, σ_{1^k} is the class of all k-planes contained in a given $n-1$-plane H. Then $\sigma^{n-k}_{1^k}$ is the class of k-planes contained in the intersection of $n-k$ general $n-1$ planes. Since $\bigcap_{i=1}^{n-k} H_i = n-(n-k) = k$, we get a unique k-plane again, so the class is $\sigma_{(n-k)^k}$.

An alternate way to think of the dimension of $G(2,n)$: Consider the Schubert cycle $\Sigma_1(\mathcal{V})$ for some flag $\mathcal{V}$. In $\mathbb{P}^{n-1}$, this Schubert cycle represents the space of lines that touch a fixed $n-3$-plane V. The dimension of this space must be $(n-2)+(n-3)$-dimensional because $n-2$ is the dimension of lines through a fixed point of $\mathbb{P}^{n-1}$ (the point of contact with V) and $n-3$ is the dimension of V. (Why do lines contained in V not bring the dimension down? I guess it's because only a closed subset of lines are actually contained in V.) Thus the dimension of $\Sigma_1(\mathcal{V})$ must be $2n-5$ and since $\Sigma_1(\mathcal{V})$ is codimension 1 in $G(2,n)$, the dimension of $G(2,n)$ must be $2n-4 = 2(n-2)$.

We can then induct on k: suppose that $\dim Gr(k,n) = k(n-k)$, then we will calculate $\dim Gr(k+1,n+1)$. The Schubert cycle $\Sigma_1(\mathcal{V})$ represents the space of k-planes in $\mathbb{P}^n$ that contact a given $n-k-1$-plane at a point. The space of k-planes touching a given point in $\mathbb{P}^n$ is $Gr(k,n)$, which has dimension $k(n-k)$ by the induction hypothesis. So the dimension of $\Sigma_1(\mathcal{V})$ must be $k(n-k)+n-k-1 = kn-k^2+n-k-1$. Finally, since $\Sigma_1(\mathcal{V})$ has codimension 1 in $Gr(k+1,n+1)$, we have $\dim Gr(k+1,n+1) = kn-k^2+n-k-1+1 = (k+1)(n-k)$.

Pieri's formula. For any Schubert class $\sigma_a \in A(G)$ and any integer λ,

$$\sigma_\lambda \cdot \sigma_a = \sum_{\substack{|c|=|a|+\lambda \\ a_i \le c_i \le a_{i-1}}} \sigma_c.$$

For example, in $\mathfrak{Gr}(4,9)$,

$$\sigma_2 \cdot \sigma_{3,2,2,1} = \sigma_{3,3,2,2} + \sigma_{5,2,2,1} + \sigma_{4,3,2,1} + \sigma_{4,2,2,2}.$$

Giambelli's formula.

$$\sigma_{a_1,a_2,\ldots,a_k} = \begin{vmatrix} \sigma_{a_1} & \sigma_{a_1+1} & \sigma_{a_1+2} & \cdots & \sigma_{a_1+k-1} \\ \sigma_{a_2-1} & \sigma_{a_2} & \sigma_{a_2+1} & \cdots & \sigma_{a_2+k-2} \\ \sigma_{a_3-2} & \sigma_{a_3-1} & \sigma_{a_3} & \cdots & \sigma_{a_3+k-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sigma_{a_k-k+1} & \sigma_{a_k-k+2} & \sigma_{a_k-k+3} & \cdots & \sigma_{a_k} \end{vmatrix}.$$

For example,

$$\sigma_{2,1} = \begin{vmatrix} \sigma_2 & \sigma_3 \\ \sigma_0 & \sigma_1 \end{vmatrix} = \sigma_2\sigma_1 - \sigma_3.$$

Another example,

$$\sigma_{3,2,2,1} = \begin{vmatrix} \sigma_3 & \sigma_4 & \sigma_5 & \sigma_6 \\ \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 \\ \sigma_0 & \sigma_1 & \sigma_2 & \sigma_3 \\ \sigma_{-2} & \sigma_{-1} & \sigma_0 & \sigma_1 \end{vmatrix} = \begin{vmatrix} \sigma_3 & \sigma_4 & \sigma_5 & 0 \\ \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 \\ 1 & \sigma_1 & \sigma_2 & \sigma_3 \\ 0 & 0 & 1 & \sigma_1 \end{vmatrix}$$
$$= \sigma_1^3\sigma_5 - \sigma_1^2\sigma_2\sigma_4 - \sigma_1^2\sigma_3^2 + \sigma_1\sigma_2^2\sigma_3 + 2\sigma_1\sigma_2\sigma_4 - \sigma_2\sigma_3^2.$$

To prove Pieri's and Giambelli's formulae, we will need some background.

Definition Schubert Dual-bert. Given a decreasing sequence $a = (a_1, \ldots, a_k)$ serving as a Schubert index, we will define the **dual index** $a^* = (n-k-a_k, \ldots, n-k-a_1)$.

Definition Transverse Flags. We say that a pair of flags $\mathcal{V}$ and $\mathcal{W}$ are **transverse** if any of the following equivalent conditions hold:

1. $V_i \cap W_{n-i} = 0$ for all i.

2. There exists a basis $e_1, \ldots, e_n$ for V in terms of which

$$V_i = \langle e_1, \ldots, e_i \rangle \text{ and } W_j = \langle e_{n-(j-1)}, \ldots, e_n \rangle.$$

Example 1. Let $\mathcal{V}$ be the flag in $\mathbb{C}^5$ given by the ordered basis

$$\{(1,0,0,0,0), (0,1,0,0,0), (0,0,1,0,0), (0,0,0,1,0), (0,0,0,0,1)\} = \{v_i\}_{i=1}^5.$$

Let $\mathcal{W}$ be the flag in $\mathbb{C}^5$ given by the ordered basis

$$\{(1,2,3,4,5), (5,6,7,8,9), (3,0,6,0,9), (2,5,2,0,1), (3,1,1,1,1,1)\} = \{w_i\}_{i=1}^5.$$

First note that $\mathcal{V}$ and $\mathcal{W}$ are in fact flags; this is clear with $\mathcal{V}$, and for $\mathcal{W}$ we can confirm

$$\begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 6 & 7 & 8 & 9 \\ 3 & 0 & 6 & 0 & 9 \\ 2 & 5 & 2 & 0 & 1 \\ 3 & 1 & 1 & 1 & 1 \end{vmatrix} = 408 \neq 0.$$

Now we will show that $\mathcal{V}$ and $\mathcal{W}$ are transverse. Following the first criterion, this is equivalent to showing

$$\{v_1, w_1, w_2, w_3, w_4\},$$
$$\{v_1, v_2, w_1, w_2, w_3\},$$
$$\{v_1, v_2, v_3, w_1, w_2\},$$
$$\{v_1, v_2, v_3, v_4, w_1\}$$

are all bases for $\mathbb{C}^5$. A quick calculation confirms that the determinants are all nonzero, so they are all bases. Therefore $\mathcal{V}$ and $\mathcal{W}$ are transverse flags.

Let's now construct a basis $\{e_1, \ldots, e_5\}$ for $\mathbb{C}^5$ such that

$$V_i = \langle e_1, \ldots, e_i \rangle \text{ and } W_j = \langle e_{n-(j-1)}, \ldots, e_n \rangle.$$

Because of how $\mathcal{V}$ is constructed, we know that

$$e_1 = (\alpha_1, 0, 0, 0, 0)$$
$$e_2 = (\alpha_2, \beta_2, 0, 0, 0)$$
$$e_3 = (\alpha_3, \beta_3, \gamma_3, 0, 0)$$
$$e_4 = (\alpha_4, \beta_4, \gamma_4, \delta_4, 0)$$
$$e_5 = (\alpha_5, \beta_5, \gamma_5, \delta_5, \varepsilon_5)$$

Where $\alpha_1, \beta_2, \gamma_3, \delta_4, \varepsilon_5 \neq 0$. Because we know $W_1 = \langle (1, 2, 3, 4, 5) \rangle$, we can choose $e_5 = (1, 2, 3, 4, 5)$. To make e_4, we need to satisfy $\langle e_4, (1, 2, 3, 4, 5) \rangle = \langle (5, 6, 7, 8, 9), (1, 2, 3, 4, 5) \rangle$. To do this, we can set e_4 equal to a linear combination of e_5 and w_2 with 0 in the fifth coordinate: for example

$$5w_2 - 9e_5 = (16, 12, 8, 4, 0) \sim (4, 3, 2, 1, 0) = e_4.$$

Now we can use e_4, e_5, and w_3 to find e_3: we know that $\langle e_3, e_4, e_5 \rangle = \langle w_1, w_2, w_3 \rangle$, so we can annihilate the fifth coordinate of w_3 by the linear combination

$$5w_3 - 9e_5 = (6, -18, 3, -16, 0)$$

and then annihilate the fourth coordinate by the linear combination

$$(6, -18, 3, -16, 0) + 16e_4 = (70, 30, 35, 0, 0) \sim (14, 6, 7, 0, 0) = e_3.$$

Now we can use e_5, e_4, e_3, and w_4 to find e_2: we know that $\langle e_2, e_3, e_4, e_5 \rangle = \langle w_1, w_2, w_3, w_4 \rangle = \langle e$, so we can annihilate the fifth coordinate of w_4 by the linear combination

$$5w_4 - e_5 = (9, 23, 7, -4, 0)$$

and then annihilate the fourth coordinate by

$$(9, 23, 7, -4, 0) + 4e_4 = (25, 35, 15, 0, 0) \sim (5, 7, 3, 0, 0)$$

and finally annihilate the third coordinate by

$$7(5,7,3,0,0) - 3e_3 = (-7,31,0,0,0) = e_2.$$

Then e_1 can still be $v_1 = (1,0,0,0,0)$. Essentially, we performed row operations (except for swapping rows) to turn the matrix

$$\begin{pmatrix} 3 & 1 & 1 & 1 & 1 \\ 2 & 5 & 2 & 0 & 1 \\ 3 & 0 & 6 & 0 & 9 \\ 5 & 6 & 7 & 8 & 9 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix} \text{ into the lower-triangular matrix } \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -7 & 31 & 0 & 0 & 0 \\ 14 & 6 & 7 & 0 & 0 \\ 4 & 3 & 2 & 1 & 0 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}.$$

Essentially, use row operations (EXCEPT ROW SWAPS) to get one invertible matrix into a lower-triangular matrix.

Example 2. For a non-example of a pair of transverse flags, pick once again $\mathcal{V}$ coming from the ordered basis

$$\{(1,0,0,0,0),(0,1,0,0,0),(0,0,1,0,0),(0,0,0,1,0),(0,0,0,0,1)\} = \{v_i\}_{i=1}^5$$

and now pick $\mathcal{W}$ as coming from the ordered basis

$$\{(1,2,3,4,5),(5,6,7,8,9),(7,8,10,12,14),(0,0,0,1,0),(0,0,0,0,1)\} = \{w_1\}_{i=1}^5.$$

First we confirm that $\{w_i\}_{i=1}^5$ is in fact a basis (the determinant is -4). Now note that $\{v_1, v_2, w_1, w_2, w_3\}$ is NOT a basis for $\mathbb{C}^5$, so $\mathcal{V}$ and $\mathcal{W}$ are not transverse. We can also see that the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 7 & 8 & 10 & 12 & 14 \\ 5 & 6 & 7 & 8 & 9 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}$$

cannot be made lower-triangular via non-swap row operations. Using the row operations $r_4 \to 5r_4 - 9r_5$, $r_3 \to 5r_3 - 14r_5$, and $r_1 \to 5r_1 - r_5$, we get

$$\begin{pmatrix} -1 & -2 & -3 & -4 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 21 & 6 & 8 & 4 & 0 \\ 16 & 12 & 8 & 4 & 0 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}.$$

Then using $r_3 \to r_3 - r_4$ and $r_2 \to 4r_2 - r_4$ and $r_1 \to r_1 + r_4$ we get

$$\begin{pmatrix} 15 & 10 & 5 & 0 & 0 \\ -16 & -12 & -8 & 0 & 0 \\ 5 & -6 & 0 & 0 & 0 \\ 16 & 12 & 8 & 4 & 0 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}.$$

Notice now that the entry in the third row and third column is 0, so we will need to swap rows to continue making this lower-triangular.

For example, if we swap rows $r_2 \leftrightarrow r_3$ we get the matrix

$$\begin{pmatrix} 15 & 10 & 5 & 0 & 0 \\ 5 & -6 & 0 & 0 & 0 \\ -16 & -12 & -8 & 0 & 0 \\ 16 & 12 & 8 & 4 & 0 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}$$

and can then use the operations $r_1 \to 8r_1 + 5r_2$ to get

$$\begin{pmatrix} 40 & 20 & 0 & 0 & 0 \\ 5 & -6 & 0 & 0 & 0 \\ -16 & -12 & -8 & 0 & 0 \\ 16 & 12 & 8 & 4 & 0 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}.$$

Finally, we an use the row operation $r_1 \to 6r_1 + 20r_2$ to get the lower-triangular matrix

$$\begin{pmatrix} 340 & 0 & 0 & 0 & 0 \\ 5 & -6 & 0 & 0 & 0 \\ -16 & -12 & -8 & 0 & 0 \\ 16 & 12 & 8 & 4 & 0 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}.$$

What this demonstrates is that the ordered basis

$$\{(1,2,3,4,5),(5,6,7,8,9),(0,0,0,1,0),(7,8,10,12,14),(0,0,0,0,1)\}$$

induces a flag transversal with $\mathcal{V}$, given by taking the original ordered basis for $\mathcal{W}$ and swapping the vectors w_3 and w_4.

We will now prove that the two definitions of transerval flags are actually equivalent.

Proof. First assume that there is a basis $\{e_1, \ldots, e_n\}$ for $\mathbb{C}^n$ where $V_i = \langle e_1, \ldots, e_i \rangle$ and $W_j = \langle e_{n-(j-1)}, \ldots, e_n \rangle$. Then $\dim(V_i \cap W_{n-i}) = \dim(\langle e_1, \ldots, e_i \rangle \cap \langle e_{i+1}, \ldots, e_n \rangle) = 0$.

Now suppose $\dim(V_i \cap W_{n-i}) = 0$ for all i, where $\mathcal{V}$ is given by the ordered basis $\{v_1, \ldots, v_n\}$ and $\mathcal{W}$ is given by the ordered basis $\{w_1, \ldots, w_n\}$. Assume without loss of generality that $\mathcal{V}$ is the standard flag (that is, the flag induced by the standard ordered basis on $\mathbb{C}^n$). Then $V_i \oplus W_{n-i} = \mathbb{C}^n$, so

$$\det \begin{pmatrix} v_1 \\ \vdots \\ v_i \\ w_{n-i} \\ \vdots \\ w_1 \end{pmatrix} \neq 0$$

for all $1 \leq i \leq n-1$. Then since

$$\det \begin{pmatrix} v_1 \\ \vdots \\ v_{n-1} \\ w_1 \end{pmatrix} \neq 0,$$

so $w_1(n) \neq 0$. Then since

$$\det \begin{pmatrix} v_1 \\ \vdots \\ v_{n-2} \\ w_2 \\ w_1 \end{pmatrix} \neq 0,$$

we know that after the row operation $r_{n-1} \to w_1(n) r_{n-1} - w_{n-1}(n) r_n$ that annihilates the n^{th} coordinate of r_{n-1}, the $(n-1)^{\text{th}}$ coordinate of r_{n-1} must be nonzero. Otherwise, the row r_{n-1} would be a linear combination of rows $r_1, \ldots, r_{n-2} = v_1, \ldots, v_{n-2}$, which is a contradiction.

Continuing, since for some i we can use row operations to turn

$$\begin{pmatrix} v_1 \\ \vdots \\ v_i \\ w_{n-i} \\ \vdots \\ w_1 \end{pmatrix} \text{ into } \begin{pmatrix} v_1 \\ \vdots \\ v_i \\ e_{i+1} \\ \vdots \\ w_1 = e_n \end{pmatrix}$$

where $e_j(k) = 0$ for all $i+1 \leq j \leq n$ and $j+1 \leq k \leq n$. Then because

$$\det \begin{pmatrix} v_1 \\ \vdots \\ v_{i-1} \\ w_{n-i+1} \\ \vdots \\ w_1 \end{pmatrix} \neq 0$$

and we can use row operations to turn

$$\begin{pmatrix} v_1 \\ \vdots \\ v_{i-1} \\ w_{n-i+1} \\ w_{n-i} \\ \vdots \\ w_1 \end{pmatrix} \text{ into } \begin{pmatrix} v_1 \\ \vdots \\ v_{i-1} \\ w_{n-i+1} \\ e_{i+1} \\ \vdots \\ e_n \end{pmatrix},$$

we can see that using row operations to annihilate the $> i$-coordinates of r_i will result in an i^{th} row that necessarily will have a nonzero entry in the i^{th} coordinate (otherwise it would be a linear combination of the above rows, which is a contradiction). This new row will be called e_i.

Continuing this way, we can see that we can construct the basis $\{e_1, \ldots, e_n\}$ by using these row operations to make a lower-triangular matrix L, which necessarily will have nonzero entries along the main diagonal. Reading the rows of L top-to-bottom yields the standard flag $\mathcal{V}$ and reading the rows bottom-to-top yields the flag $\mathcal{W}$ because each row r_j of L is a linear combination of the rows underneath with vectors from W_j.

If $\mathcal{V}$ is not the standard flag, we can perform a change of basis B to make $B\mathcal{V}$ the standard flag, and then $B\mathcal{W}$ is transerval to $B\mathcal{V}$ if and only if $\mathcal{W}$ is transversal to $\mathcal{V}$. So there exists a basis $\{e_1, \ldots, e_n\}$ on $\mathbb{C}^n$ such that $BV_i = \langle e_1, \ldots e_i\rangle$ and $BW_j = \langle e_{n-(j-1)}, \ldots, e_n\rangle$, so $\{B^{-1}e_1, \ldots, B^{-1}e_n\}$ is a basis for $\mathbb{C}^n$ such that $V_i = \langle B^{-1}e_1, \ldots, B^{-1}e_i\rangle$ and $W_j = \langle B^{-1}e_{n-(j-1)}, \ldots, B^{-1}e_n\rangle$. Thus we have proven the equivalence. □

Note the two transverse pairs may be carried to each other by a linear automorphism of V. Moreover, transverse pairs form a dense open subset in the space of all pairs of flags, so any statement proves for a general pair of flags (such as the general transversality of the intersection $\Sigma_a(\mathcal{V}) \cap \Sigma_b(\mathcal{W}) \subseteq G$) holds for any transverse pair, and vice versa.

Proposition 4.6. If $\mathcal{V}$ and $\mathcal{W}$ are transverse flags in V and $\Sigma_a(\mathcal{V})$ and $\Sigma_b(\mathcal{W})$ are Schubert cycles with $|a| + |b| = k(n-k)$, then $\Sigma_a(\mathcal{V})$ and $\Sigma_b(\mathcal{W})$ intersect transversely at a unique point if $b = a^*$ and are disjoint otherwise.

Proof. Since the two flags $\mathcal{V}$ and $\mathcal{W}$ are transverse, the Schubert cycles will meet generically transversely, and hence (since the intersection is zero-dimensional) transversely. Thus

$$\begin{aligned}\deg \sigma_a\sigma_b &= \#(\Sigma_a(\mathcal{V}) \cap \Sigma_b(\mathcal{W})) \\ &= \#\left\{\Lambda : \begin{array}{c}\dim(V_{n-k+i-a_i} \cap \Lambda) \geq i, \\ \dim(W_{n-k+i-b_i} \cap \Lambda) \geq i,\end{array} \text{for all } i\right\}.\end{aligned}$$

To evaluate the cardinality of this set, consider the conditions in pairs: that is, for each i, consider the i^{th} condition associated to the Schubert cycles $\Sigma_a(\mathcal{V})$:

$$\dim(V_{n-k+i-a_i}) \cap \Lambda) \geq i$$

in combination with the $(k-i+1)^{\text{th}}$ condition assocated to $\Sigma_b(\mathcal{W})$:

$$\dim(W_{n-i+1-b_{k-i+1}} \cap \Lambda) \geq k-i+1.$$

If these conditions are both satisfied, then the subspaces

$$V_{n-k+i-a_i} \cap \Lambda \text{ and } W_{n-i+1-b_{k-i+1}} \cap \Lambda,$$

having greather than complementary dimension in Λ (Λ is k-dimensional), must have nonzero intersection; in particular, we must have

$$V_{n-k+i-a_i} \cap W_{n-i+1-b_{k-i+1}} \neq 0,$$

and since the flags $\mathcal{V}$ and $\mathcal{W}$ are general, this in turn says that we must have

$$n-k+i-a_i+n-i+1-b_{k-i+1} \geq n+1,$$

or in other words

$$a_i+b_{k-i+1} \leq n-k$$

or

$$(k-i+a_i)+(i-1+b_{k-i+1}) \leq n-1.$$

If equality holds in this last inequality, the subspaces $V_{n-k+i-a_i}$ and $W_{n-i+1-b_{k-i+1}}$ will meet in a one-dimensional vector space Γ_i, necessarily contained in Λ. (This last point is easier to understand if you look back to the long inequality, rather than the simplified one. Or even the last inequality, which would say that the sum of the codimensions is $n-1$, so the general vector spaces meet at a line.)

We have thus seen that $\Sigma_a(\mathcal{V})$ and $\Sigma_b(\mathcal{W})$ will be disjoint unless $a_i+b_{k-i+1} \leq n-k$ for all i. But from the equality

$$|a|+|b| = \sum_{i=1}^{k}(a_i+b_{k-i+1}) = k(n-k),$$

we see that if $a_i+b_{k-i+1} \leq n-k$ for all i, then we must have $a_i+b_{k-i+1} = n-k$ for all i. Moreover, in this case any Λ in the intersection $\Sigma_a(\mathcal{V}) \cap \Sigma_b(\mathcal{W})$ must contain each of the k subspaces Γ_i, so there is a unique such Λ, equal to the span of these one-dimensional spaces, as required. □

We now get an approach to determining the coefficients in the expression of the class of a cycle as a linear combination of Schubert classes: if $\Gamma \subseteq G$ is any cycle of pure codimension m, we can write

$$[\Gamma] = \sum_{|a|=m} \gamma_a \sigma_a.$$

To find the coefficient γ_a, we intersect both sides with the Schubert cycle $\Sigma_{a^*}(\mathcal{V}) = \Sigma_{n-k-a_k,\ldots,n-k-a_i}(\mathcal{V})$ for a general flag $\mathcal{V}$; we then have

$$\gamma_a = \deg([\Gamma] \cdot \sigma_{a^*} = \#(\Gamma \cap \Sigma_{a^*}(\mathcal{V})).$$

This is the method of undetermined coefficients. Explicitly, we have:

Corollary 4.8. If $\alpha \in A^m(G)$ is any class, then

$$\alpha = \sum_{|a|=m} \deg(\alpha\sigma_{a^*}) \cdot \sigma_a.$$

In particular, if σ_a and $\sigma_b \in A(G)$ are any Schubert classes on $G = G(k,n)$, then the product $\sigma_a\sigma_b$ is equal to

$$\sum_{|c|=|a|+|b|} \gamma_{a,b;c}\sigma_c,$$

where

$$\gamma_{a,b;c} = \deg(\sigma_a\sigma_b\sigma_{c^*}).$$

We are now ready to prove Pieri's formula.

Proof. By Corollary 4.8, Pieri's formula is equivalent to the assertion that, for any Schubert index c with $|c| = |a| + \lambda$,

$$\deg(\sigma_a \sigma_\lambda \sigma_{c^*}) = \begin{cases} 1 & \text{if } a_i \leq c_i \leq a_{i-1} \text{ for all } i \\ 0 & \text{otherwise} \end{cases}.$$

To prove this, we will look at the corresponding Schubert cycles $\Sigma_a(\mathcal{V})$, $\Sigma_\lambda(\mathcal{U})$ and $\Sigma_{c^*}(\mathcal{W})$, defined with respect to general flags; we will show their intersection is empty if c_i violates the condition $a_i \leq c_i \leq a_{i-1}$ for any i, and consists of a single point otherwise. Since general flags are transverse, the intersection multiplicity will be 1 in the latter case.

By definition,

$$\Sigma_a(\mathcal{V}) = \{\Lambda : \dim(\Lambda \cap V_{n-k+i-a_i}) \geq i \text{ for all } i\}$$

and

$$\Sigma_{c^*}(\mathcal{W}) = \{\Lambda : \dim(\Lambda \cap W_{i+c_{k+1-i}}) \geq i \text{ for all } i\}.$$

Set

$$A_i = V_{n-k+i-a_i} \cap W_{k+1-i+c_i},$$

so that either $A_i = 0$ or $\dim A_i = c_i - a_i + 1$. (NOTE: The dimensions of $V_{n-k+i-a_i}$ and $W_{k+1-i+c_i}$ add up to $n + 1 + c_i - a_i$, so $\dim A_i = 0$ if $c_i - a_i + 1 \leq 0$ or $\dim A_i = c_i - a_i + 1$ if $c_i - a_i + 1 > 0$.) Combining the i^{th} condition in the first definition and the $(k + 1 - i)^{\text{th}}$ condition in the second, we see that for any $\Lambda \in \Sigma_a(\mathcal{V}) \cap \Sigma_{c^*}(\mathcal{W})$ we have

$$\Lambda \cap A_i \neq 0,$$

because $\dim(B_i = \Lambda \cap V_{n-k+i-a_i}) \geq i$ and $\dim(C_i = \Lambda \cap W_{k+1-i+c_i}) \geq k+1-i$. So B_i and C_i are subspaces of Λ whose dimensions add to $k+1 = \dim(\Lambda)+1$, so they must have nontrivial intersection within Λ. If $c_i < a_i$ for some i then $A_i = 0$ so that $\Sigma_a(\mathcal{V}) \cap \Sigma_{c^*}(\mathcal{W}) = \varnothing$, and $\deg(\sigma_a \sigma_\lambda \sigma_{c^*}) = 0$, as required. Thus we may assume that $c_i \geq a_i$ for every i.

We claim that the A_i are linearly independent if and only if $c_i \leq a_{i-1}$ for all i. To see this, choose a basis e_i so that $V_i = \langle e_1, \ldots, e_i \rangle$ and $W = \langle e_{n-j+1}, \ldots, e_n \rangle$. Then

$$A_i = \langle e_{n-k_i-c_i}, \ldots, e_{n-k+i-a_i} \rangle,$$

and the condition $c_i \leq a_{i-1}$ amounts to the condition that the two successive ranges of indices $n-k+i-1-c_{i-1}, \ldots, n-k+i-1-a_{i-1}$ and $n-k+i-c_i, \ldots, n-k+i-a_i$ do not overlap. In other words, if we let

$$A = \langle A_1, \ldots, A_k \rangle$$

be the span of the spaces A_i, then we have

$$\dim A \leq \sum c_i - a_i + 1 - k + \lambda,$$

with equality holding if and only if $c_i \leq a_{i-1}$ for all i.

Now we introduce the conditions associated with the special Schubert cycle $\Sigma_\lambda(\mathcal{U})$. this is the set of k-planes that have nonzero intersection with a general linear subspace $U = U_{n-k+1-\lambda} \subseteq V$. For there to be any $\Lambda \in \Sigma_a(\mathcal{V}) \cap \Sigma_{c^*}(\mathcal{W})$ satisfying this additional condition requires that $A \cap U \neq 0$, and hence, since U is general, that $\dim A \geq k + \lambda$. Thus, if $c_i \geq a_{i-1}$ for any i, then we will have $\Sigma_a(\mathcal{V}) \cap \Sigma_{c^*}(\mathcal{W}) \cap \Sigma_\lambda(\mathcal{U}) = \varnothing$. We can accordingly assume $c_i \leq a_{i-1}$ for all i, and hence $\dim A = k + \lambda$.

Finally, since $U \subseteq V$ is a general subspace of codimension $k + \lambda - 1$, it will meet A in a one-dimensional subspace. Choose any nonzero vector v in this intersection. Since $A = \bigoplus A_i$, we can write v uniquely as a sum

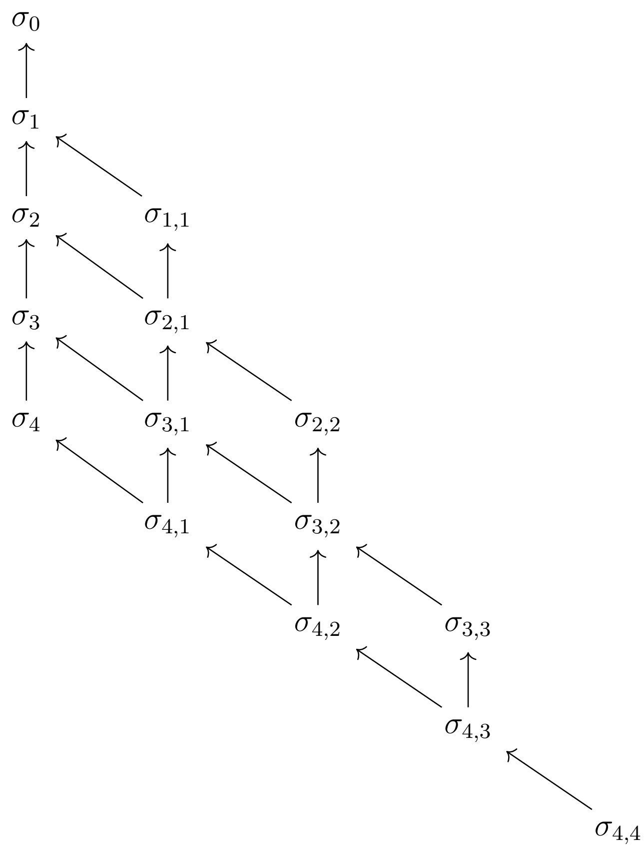
$$v = v_1 + \cdots + v_k \text{ with } v_i \in A_i.$$

Suppose now that $\Lambda \in \Sigma_a(\mathcal{V}) \cap \Sigma_\lambda(\mathcal{U}) \cap \Sigma_{c^*}(\mathcal{W})$ satisfies all the Schubert conditions above. Since $\Lambda \subseteq A$ and $\Lambda \cap U \neq 0$, Λ must contain the vector v, and since Λ is spanned by its intersections with the A_i, it follows that Λ must contain the vectors v_i as well. Thus, we see that the intersection $\Sigma_a(\mathcal{V}) \cap \Sigma_\lambda(\mathcal{U}) \cap \Sigma_{c^*}(\mathcal{W})$ will consist of the single point corresponding to the plane $\Lambda = \langle v_1, \ldots, v_k \rangle$ spanned by the v_i, and we are done. □

We will now answer the following question: What is the degree of the Grassmannian $G(2, n+1)$ under the Plücker embedding? We observe first that, since the hyperplane class on $\mathbb{P}(\bigwedge^2 k^{n+1})$ pulls back to the class $\sigma_1 \in A^1(G(2, n+1))$, we have

$$\deg(G(2, n+1)) = \deg(\sigma_1^{2n-2}).$$

Recall that $\dim G(2, n+1) = 2(n+1-2) = 2n-2$. To evaluate this product, we make a directed graph with the Schubert classes σ_a in $G(2, n+1)$ as vertices and with the inclusions among the corresponding Schubert cycles $\Sigma_a(\mathcal{V})$ indicated by arrows (the graph shown is the case $n = 5$):



In terms of this graph, the rule for multiplication is simple: The product of any Schubert class $\sigma_{a,b}$ with σ_1 is the sum of all immediate predecessors of $\sigma_{a,b}$– that is, the Schubert classes in the row below $\sigma_{a,b}$ that are connected to $\sigma_{a,b}$ by an arrow. In particular, the degree $\deg((\sigma_1)^{2n-2})$ of the Grassmannian is the number of paths upward through this diagram starting with $\sigma_{n-1,n-1}$ (the bottom) and ending with $\sigma_{0,0}$ (the top). If we designate such a path by a sequence of $n-1$ 1's and $n-1$ 2's, corresponding to whether the first or second index changes, reading from left to right, there are never more 1's than 2's. Equivalently, if we use left and right parentheses for 2's and 1's respectively, this is the number of ways in which $n-1$ pairs of parentheses can appear in a grammatically correct sentence. This is the $(n-1)^{\text{th}}$ *Catalan number*; in combinatorics it is known that

$$c_{n-1} = \frac{(2n-2)!}{n!(n-1)!}.$$

So the degree of the Grassmannian $G(2, n+1) \subseteq \mathbb{P}(\bigwedge^2 k^{n+1})$ is $\dfrac{(2n-2)!}{n!(n-1)!}$, which is also the number of lines in $\mathbb{P}^n$ that meet $2n-2$ general $n-2$-planes in $\mathbb{P}^n$.

Note that σ_1 is the class of the hyperplane section of any Grassmannian under the Plücker embedding. With the aid of the hook formula from combinatorics, we can work out

$$\deg(G(k,n)) = (k(n-k))! \prod_{i=0}^{k-1} \frac{i!}{(n-k+i)!}.$$

Now on to Chern classes. Recall the Chern classes of a vector bundle can be computed as the degeneracy loci of the global sections (that is, Chern classes measure the extent to which a vector bundle is nontrivial, with a trivial vector bundle just being the product of a manifold with a vector space). But there are other ways of building the Chern classes when the bundle is built from simpler bundles.

Example: 27 lines on a cubic surface. Given a smooth cubic surface $X \subseteq \mathbb{P}^3$ determined by the vanishing of a cubic form F in four variables, we wish to determine the degree of the locus in $\mathrm{PGr}(1,3) = \mathfrak{Gr}(2,4)$ of lines contained in X. We linearize the problem using the observation that, if we fix a particular line $L \subseteq \mathbb{P}^3$, then the condition that L lies on X can be expressed as four linear conditions on the coefficients of F: to see this, note that the restriction map from the 20-dimensional vector space of cubic forms in $\mathbb{P}^3$ to the four-dimensional vector space $V_L = H^0(\mathcal{O}_L(3))$ of cubic forms on a line $L \cong \mathbb{P}^1 \subseteq \mathbb{P}^3$ (four dimensions are x^3, x^2y, xy^2, and y^3) is a linear surjection, and the condition for the inclusion $L \subseteq X$ is that F maps to 0 in V_L.

As the line L varies over $\mathfrak{Gr}(2,4)$, the four-dimensional spaces V_L of cubic forms on the varying lines L fit together to form a vector bundle $\mathcal{V}$ of rank 4 on $\mathfrak{Gr}(2,4)$. A cubic form F on $\mathbb{P}^3$, through its restriction to each V_L, defines an algebraic global section σ_F of this vector bundle. Thus the locus of lines contained in the cubic surface X is the zero locus of the section σ_F. Assuming for the moment that this zero locus is zero-dimensional, we call its class in $A(\mathfrak{Gr}(2,4))$ the *fourth Chern class* of $\mathcal{V}$, denoted $c_4(\mathcal{V})$.

We can build $\mathcal{V}$ by first examining the rank-2 vector bundle $\mathcal{S}^*$ on $\mathfrak{Gr}(2,4)$ by $\mathcal{S}^*_L = H^0(\mathcal{O}_L(1))$ consisting of linear functions on L. Then the Chern class of S^*_L reflects the number of lines on planes instead of cubic surfaces. Given a linear form H on $\mathbb{P}^3$ one obtains a section σ_H of S^*_L by $\sigma_H(L) = H_L$. The zero locus of σ_H is simply the Schubert cycle $\Sigma_{1,1}(H) = \begin{pmatrix} * & * & 0 & 0 \\ * & * & * & 0 \end{pmatrix}$, lines that are contained in the plane H. Thus $c_2(\mathcal{S}^*) = \sigma_{1,1}$. Similarly, given two linear forms H_1 and H_2 on $\mathbb{P}^3$, then σ_{H_1} and σ_{H_2} are linearly dependent if and only if $L \cap (H_1 \cap H_2) \neq \varnothing$. So $c_1(\mathcal{S}^*) = \sigma_1$, the class of lines touching a given line. Our remaining task is to relate $c(\mathcal{V})$ and $c(\mathcal{S}^*)$ using $\mathcal{V} = \mathrm{Sym}^3\mathcal{S}^*$.

We must first see that we can build $\mathcal{S}^*$ as the direct product of two line bundles, $\mathcal{S}^* = L \oplus M$. Then write $c_1(L) = \alpha$ and $c_1(M) = \beta$, so $c(L) = 1+\alpha$ and $c(M) = 1+\beta$. Then $c(\mathcal{S}^*) = (1+\alpha)(1+\beta)$, meaning $c_1(\mathcal{S}^*) = \alpha+\beta$ and $c_2(\mathcal{S}^*) = \alpha\beta$. Note that

$$\mathrm{Sym}^2\mathcal{S}^* = L^2 \oplus L \otimes M \oplus M^2$$

and so

$$c(\mathrm{Sym}^2\mathcal{S}^*) = (1+2\alpha)(1+\alpha+\beta)(1+2\beta) = 1 + 3(\alpha+\beta) + (2\alpha^2+2\beta^2+8\alpha\beta) + 4\alpha\beta(\alpha+\beta),$$

which we can express in terms of $c_1(\mathcal{S}^*)$ and $c_2(\mathcal{S}^*)$ as

$$1 + 3c_1(\mathcal{S}^*) + (2c_1^2(\mathcal{S}^*) + 4c_2(\mathcal{S}^*)) + 4c_2(\mathcal{S}^*)c_1(\mathcal{S}^*).$$

This formula holds for any rank 2 vector bundle. So in particular, we see $c_1(\mathrm{Sym}^2\mathcal{S}^*) = 3c_1(\mathcal{S}^*)$, $c_2(\mathrm{Sym}^2\mathcal{S}^*) = 2c_1^2(\mathcal{S}^*) + 4c_2(\mathcal{S}^*)$, and $c_3(\mathrm{Sym}^2\mathcal{S}^*) = 4c_2(\mathcal{S}^*)c_1(\mathcal{S}^*)$. Moreover,

$$\mathrm{Sym}^3(\mathcal{S}^*) = L^3 \oplus L^2 \otimes M \oplus L \otimes M^2 \oplus M^3$$

yields

$$\begin{aligned} c_4(\mathrm{Sym}^3\mathcal{S}^*) &= [(1+3\alpha)(1+2\alpha+\beta)(1+\alpha+2\beta)(1+3\beta)]_{\deg 4} \\ &= 3\alpha(2\alpha+\beta)(\alpha+2\beta)3\beta = 9c_2(\mathcal{S}^*)(2c_1^2(\mathcal{S}^*)+c_2(\mathcal{S}^*)). \end{aligned}$$

Recalling that in our example $c_1(\mathcal{S}^*) = \sigma_1$ and $c_2(\mathcal{S}^*) = \sigma_{1,1}$, we have

$$c_4(\mathrm{Sym}^3\mathcal{S}^*) = 9\sigma_{1,1}(2\sigma_1^2+\sigma_{1,1}) = 9\sigma_{1,1}(2\sigma_2+3\sigma_{1,1}) = 27.$$

So we get that there are 27 lines on a cubic surface.

Another Chern class example. Consider the bundle $\mathcal{E}$ over $\mathfrak{Gr}(3,4) = \mathrm{PGr}(2,3)$ of plane conics in a plane in $\mathbb{P}^3$. That is, the fibers of the projective bundle $\mathbb{P}\mathcal{E}_H \cong \mathbb{P}^5$ are the space of the conics in H. Then $\mathcal{E}$ is a rank 6 vector bundle over $\mathrm{PGr}(2,3) = \mathbb{P}^{3*}$. Note that $\mathbb{P}\mathcal{E} = \mathbb{P}\mathrm{Sym}^2(S^*)$, the bundle of lines in a plane of $\mathbb{P}^3$. We can apply the Theorem that says

$$A(\mathbb{P}\mathcal{E}) = A(X)[\zeta]/(\zeta^r + c_1(\mathcal{E})\zeta^{r-1} + \cdots c_{r-1}(\mathcal{E})\zeta + c_r(\mathcal{E})),$$

but first we will need to find the total Chern class $c(\mathcal{E})$. We will use $A(X) = A(\mathbb{P}^{3*}) = \mathbb{Z}[\omega]/(\omega^4)$, where ω is the class of a plane in $\mathbb{P}^{3*}$ and so corresponds to the cycle of planes in $\mathbb{P}^3$ containing a given point.

Since S^* is a rank 3 vector bundle, we can write $S^* = L \oplus M \oplus N$, so

$$c(S^*) = 1+\omega+\omega^2+\omega^3 = (1+\alpha)(1+\beta)(1+\gamma).$$

Then $\mathrm{Sym}^2 S^* = L^2 \oplus L\otimes M \oplus L \otimes N \oplus M^2 \oplus M\otimes N \oplus N^2$, so

$$c(\mathrm{Sym}^2 S^*) = (1+2\alpha)(1+\alpha+\beta)(1+\alpha+\gamma)(1+2\beta)(1+\beta+\gamma)(1+2\gamma).$$

We want to put this in terms of ω. Our initial equality $1+\omega+\omega^2+\omega^3 = (1+\alpha)(1+\beta)(1+\gamma)$ gives us

- $\omega = \alpha+\beta+\gamma$
  - $\omega^2 = \alpha^2+2\alpha\beta+2\alpha\gamma+\beta^2+2\beta\gamma+\gamma^2$
  - $\omega^3 = \alpha^3+3\alpha^2\beta+3\alpha^2\gamma+3\alpha\beta^2+6\alpha\beta\gamma+3\alpha\gamma^2+\beta^3+3\beta^2\gamma+3\beta\gamma^2+\gamma^3$
- $\omega^2 = \alpha\beta+\alpha\gamma+\beta\gamma$
  - $\omega^3 = (\alpha\beta+\alpha\gamma+\beta\gamma)(\alpha+\beta+\gamma) = \alpha^2\beta+\alpha^2\gamma+\alpha\beta^2+3\alpha\beta\gamma+\alpha\gamma^2+\beta^2\gamma+\beta\gamma^2$
- $\omega^3 = \alpha\beta\gamma$

The degree ≤ 3 terms of $(1+2\alpha)(1+\alpha+\beta)(1+\alpha+\gamma)(1+2\beta)(1+\beta+\gamma)(1+2\gamma)$ are

$$\begin{aligned} &1+4\alpha+4\beta+4\gamma+5\alpha^2+15\alpha\beta+15\alpha\gamma+5\beta^2+15\beta\gamma+5\gamma^2 \\ &+2\alpha^3+17\alpha^2\beta+17\alpha^2\gamma+17\alpha\beta^2+52\alpha\beta\gamma+17\alpha\gamma^2+2\beta^3+17\beta^2\gamma+17\beta\gamma^2+2\gamma^3. \end{aligned}$$

The degree 1 component is $4(\alpha+\beta+\gamma) = 4\omega$. Note that our equations give $(\alpha^2+2\alpha\beta+2\alpha\gamma+\beta^2+2\beta\gamma+\gamma^2)-(\alpha\beta+\alpha\gamma+\beta\gamma) = \alpha^2+\alpha\beta+\alpha\gamma+\beta^2+\beta\gamma+\gamma^2 = 0$. The degree 2 component is thus

$$\begin{aligned}&5\alpha^2+15\alpha\beta+15\alpha\gamma+5\beta^2+15\beta\gamma+5\gamma^2\\ &= 5\alpha^2+15\alpha\beta+15\alpha\gamma+5\beta^2+15\beta\gamma+5\gamma^2-5(\alpha^2+\alpha\beta+\alpha\gamma+\beta^2+\beta\gamma+\gamma^2)\\ &= 10(\alpha\beta+\alpha\gamma+\beta\gamma) = 10\omega^2.\end{aligned}$$

Our equations also give us that

$$A := \alpha^3+2\alpha^2\beta+2\alpha^2\gamma+2\alpha\beta^2+3\alpha\beta\gamma+2\alpha\gamma^2+\beta^3+2\beta^2\gamma+2\beta\gamma^2+\gamma^3 = 0$$

and

$$B := \alpha^2\beta+\alpha^2\gamma+\alpha\beta^2+2\alpha\beta\gamma+\alpha\gamma^2+\beta^2\gamma+\beta\gamma^2 = 0.$$

So our degree 3 component is

$$\begin{aligned}&2\alpha^3+17\alpha^2\beta+17\alpha^2\gamma+17\alpha\beta^2+52\alpha\beta\gamma+17\alpha\gamma^2+2\beta^3+17\beta^2\gamma+17\beta\gamma^2+2\gamma^3-2A\\ &= 13\alpha^2\beta+13\alpha^2\gamma+13\alpha\beta^2+46\alpha\beta\gamma+13\alpha\gamma^2+13\beta^2\gamma+13\beta\gamma^2-13B = 20\alpha\beta\gamma = 20\omega^3.\end{aligned}$$

Therefore we have

$$\begin{aligned}c(\mathrm{Sym}^2 S^*) &= (1+2\alpha)(1+\alpha+\beta)(1+\alpha+\gamma)(1+2\beta)(1+\beta+\gamma)(1+2\gamma)\\ &= 1+4\omega+10\omega^2+20\omega^3.\end{aligned}$$

You can also do this whole calculation using e.g. Macaulay2 by inputting the ring

$$\mathbb{Z}[\alpha,\beta,\gamma]/((\alpha+\beta+\gamma)^2-(\alpha\beta+\alpha\gamma+\beta\gamma),(\alpha+\beta+\gamma)^3-\alpha\beta\gamma)$$

and simplifying the polynomial $c(\mathrm{Sym}^2 S^*)$.

Therefore we have

$$A(\mathcal{E}) = \mathbb{Z}[\omega,\zeta]/(\omega^4, \zeta^6+4\omega\zeta^5+10\omega^2\zeta^4+20\omega^3\zeta^3)$$

where ζ represents the class that restricts to a hyperplane $\mathbb{P}^4$ on each fiber ω^3 (note that a general $\mathbb{P}^4$ of conics is basepoint-free, so ζ does not restrict to the class of conics containing a given point). That is to say, $\omega^3\zeta$ is a $\mathbb{P}^4$ of conics in the plane represented by ω^3, while ω^3 is itself the fiber class: a full $\mathbb{P}^5$ of conics in the given plane.

Now consider the class δ of conics intersecting a line in $\mathbb{P}^3$. We want to determine $\deg(\delta^8)$.

First use undetermined coefficients to find $\delta = p\omega + q\zeta \in A^1(\mathcal{E})$ the class of all plane conics through a given line L and find $\deg(\delta^8)$. To this effect, find the degree of δ in each component: that is, what is $\deg(\delta\omega^2\zeta^5)$ and $\deg(\delta\omega^3\zeta^4)$? Note $\omega^2\zeta^5$ is a line in the $\mathbb{P}^{3*}$ (so it represents a pencil of planes with a "fixed" conic) and $\omega^3\zeta^4$ is a line in the $\mathbb{P}^5$ (so it represents a pencil of conics in a fixed plane).

I calculated that $\deg(\delta\omega^3\zeta^4) = 1$ because a general pencil of conics will have one fibre containing a general point (the general point where is the intersection of the general line L with the fixed plane).

I then calculated that $\deg(\delta\omega^2\zeta^5) = 2$ because the intersection of L with a general pencil of planes will trace out a line in a plane containing the fixed conic C, which will intersect C at two points.

From this we see that $\delta = 2\omega + \zeta$. Now we can calculate

$$\delta^8 = (\zeta + 2\omega)^8 = \zeta^8 + 16\zeta^7\omega + 112\zeta^6\omega^2 + 448\zeta^5\omega^3.$$

We know that $\omega^3\zeta^5 = 1$, since it is the class of a unique conic. Using

$$\zeta^6 + 4\omega\zeta^5 + 10\omega^2\zeta^4 + 20\omega^3\zeta^3 = \omega^4 = 0,$$

we can see that

$$\omega^2(\zeta^6 + 4\omega\zeta^5 + 10\omega^2\zeta^4 + 20\omega^3\zeta^3) = \omega^2\zeta^6 + 4\omega^3\zeta^5 = 0,$$

so $\deg(\omega^2\zeta^6) = -4$. Similarly,

$$\omega\zeta(\zeta^6 + 4\omega\zeta^5 + 10\omega^2\zeta^4 + 20\omega^3\zeta^3) = \omega\zeta^7 + 4\omega^2\zeta^6 + 10\omega^3\zeta^5 = 0,$$

so $\omega\zeta^7 - 16 + 10 = 0$, thus $\omega\zeta^7 = 6$. Finally,

$$\zeta^2(\zeta^6 + 4\omega\zeta^5 + 10\omega^2\zeta^4 + 20\omega^3\zeta^3) = \zeta^8 + 4\omega\zeta^7 + 10\omega^2\zeta^6 + 20\omega^3\zeta^5 = 0,$$

so $\zeta^8 + 4(6) + 10(-4) + 20(1) = 0$, giving $\zeta^8 = -4$. Putting this all together, we get

$$\delta^8 = -4 + 16(6) + 112(-4) + 448(1) = 92.$$

Therefore there are 92 plane conics in $\mathbb{P}^3$ intersecting 8 given general lines.

It is also worth mentioning that $\delta^5\omega^3 = 1$, meaning that there is a unique plane conic contained in a given plane and meets five lines in $\mathbb{P}^3$: specifically, the unique conic that meets the five points of intersection of the five lines with the plane. The computation works out: $\delta^5\omega^3 = (2\omega+\zeta)^5\omega^3 = (\zeta^5 + 10\zeta^4\omega + 40\zeta^3\omega^2 + 80\zeta^2\omega^3 + 80\zeta\omega^4 + 32\omega^5)\omega^3 = \zeta^5\omega^3 = 1$.

A similar calculation reveals $\delta^7\omega = 34$, so there are 34 plane conics through 7 given lines and are coplanar with a given point. Similarly, $\delta^6\omega^2 = 8$, so there are 8 plane conics intersecting 6 given lines and are coplanar with 2 given points (equivalently(?), conics intersecting 6 given lines and intersect a seventh given line twice).

Lines on a Quintic Threefold

Let $\mathcal{E}$ be a rank 6 vector bundle on $Gr(2,5)$ associating to each $L \in Gr(2,5)$ the 6-dimensional vector space $H^0(\mathcal{O}_L(5)) = \langle x^5, x^4y, x^3y^2, x^2y^3, xy^4, y^5\rangle$. Let Q be a quintic threefold in $\mathbb{P}^4$. Then Q gives a section on $\mathcal{E}$ by $\sigma_Q(L) = L \cap Q$ for each $L \in Gr(2,5)$. Then the locus on lines contained in Q is the zero locus of σ_Q, which is $c_6(\mathcal{E})$.

Let $\mathcal{S}$ be a rank 2 vector bundle on $Gr(2,5)$ giving a point to each line. Then $\mathcal{E} = \mathrm{Sym}^5\mathcal{S}$. We can write $\mathcal{S} = L \oplus M$ and so $c(\mathcal{S}) = (1+\alpha)(1+\beta)$ where $\alpha = c_1(L)$ and $\beta = c_1(M)$. Then $c(\mathcal{S}) = 1 + \alpha + \beta + \alpha\beta$, and so $c_1(\mathcal{S}) = \alpha + \beta$ and $c_2(\mathcal{S}) = \alpha\beta$.

Furthermore, we can write $c_2(\mathcal{S})$ in terms of Schubert classes on $Gr(2,5)$. Given a linear form $H \subseteq \mathbb{P}^4$, one obtains a section σ_H of $\mathcal{S}$ by $\sigma_H(L) = H \cap L$. The zero locus of σ_H is lines contained in H, which is the Schubert class $\sigma_{1,1}$.

Similarly, $c_1(\mathcal{S})$ can be calculated by taking two linear forms H_1 and H_2. Then σ_{H_1} and σ_{H_2} are linearly dependent if $L \cap (H_1 \cap H_2) \neq \varnothing$. So $c_1(\mathcal{S})$ is the class of lines touching a given plane, which is $\sigma_{1,0}$. Thus $\alpha + \beta = \sigma_1$ and $\alpha\beta = \sigma_{1,1}$.

Note

$$\mathcal{E} = \mathrm{Sym}^5\mathcal{S} = L^5 \oplus L^4 \otimes M \oplus L^3 \otimes M^2 \oplus L^2 \otimes M^3 \oplus L \otimes M^4 \oplus M^5$$

and so

$$c_6(\mathcal{E}) = (5\alpha)(4\alpha+\beta)(3\alpha+2\beta)(2\alpha+3\beta)(\alpha+4\beta)(5\beta)$$

which can be rewritten as

$$25\alpha\beta \cdot (4\alpha+\beta)(\alpha+4\beta) \cdot (3\alpha+2\beta)(2\alpha+3\beta).$$

This is equal to

$$25\alpha\beta \cdot [4(\alpha^2+2\alpha\beta+\beta^2)+9\alpha\beta] \cdot [6(\alpha^2+2\alpha\beta+\beta^2)+\alpha\beta]$$

which can be written into Schubert classes as

$$25\sigma_{1,1}[4\sigma_1^2+9\sigma_{1,1}][6\sigma_1^2+\sigma_{1,1}] = 25\sigma_{1,1}[4\sigma_2+13\sigma_{1,1}][6\sigma_2+7\sigma_{1,1}]$$
$$= 25\sigma_{1,1}[24\sigma_2^2+106\sigma_2\sigma_{1,1}+91\sigma_{1,1}^2] = 600\sigma_{1,1}\sigma_2^2+2650\sigma_{1,1}^2\sigma_2+2275\sigma_{1,1}^3 = 600+2275 = 2875.$$

Thus there are 2875 lines contained in a general quintic threefold in $\mathbb{P}^4$. Recall that σ_1 is the class of lines touching a given plane, $\sigma_{1,1}$ is the class of lines contained in a given hyperplane and σ_2 is the class of lines touching a given line. Thus $\sigma_{1,1}^3$ is the class of lines contained in three given hyperplanes, but the intersection of three general hyperplanes is a line, so $\sigma_{1,1}^3$ is a unique line. Similarly, $\sigma_{1,1}\sigma_2^2$ is the class of lines contained in a hyperplane H and touching two given lines $L_1, L_2 \in \mathbb{P}^4$. Thus the line must touch the points $p_1 = H \cap L_1$ and $p_2 = H \cap L_2$, so $\sigma_{1,1}\sigma_2^2$ is the class of lines containing two given points, and so is also a unique line. By contrast, $\sigma_{1,1}^2\sigma_2$ is the class of lines contained in a plane in $\mathbb{P}^4$ and touching a line in $\mathbb{P}^4$: but in general lines and planes are skew in $\mathbb{P}^4$, so this class is 0.

Now on to Shapiro-Shapiro and the hook formula. The hook formula measures the amount of young tableux or something....

The Shapiro-Shapiro conjecture deals with real solutions to enumeration puzzles. For example, recall the 2×2 minors of the matrix

$$\begin{pmatrix} a & b & c & d \\ e & f & g & h \end{pmatrix}$$

yield a Plücker embedding into $Gr(2,4)$ for the line in $\mathbb{P}^3$ that contains the points (a,b,c,d) and (e,f,g,h). The determinant of the matrix

$$\begin{pmatrix} a & b & c & d \\ e & f & g & h \\ 1 & t & t^2 & t^3 \\ 0 & 1 & 2t & 3t^2 \end{pmatrix}$$

for some parameter t can be expressed as a polynomial with variable t and whose coefficients are the variables of the Plücker embedding. This determinant is

$$\begin{aligned} ch + aht^2 - 2bht - bet^4 + 2cet^3 - det^2 + aft^4 - 3cft^2 + 2dft - dg - 2agt^3 + 3bgt^2 \\ = (ch - dg) - 2(bh - df)t + (ah - de)t^2 + 3(bg - cf)t^2 - 2(ag - ce)t^3 - (af - be)t^4 \\ = P_{2,3} - 2P_{1,3}t + (P_{0,3} + 3P_{1,2})t^2 - 2P_{0,2}t^3 - P_{0,1}t^4. \end{aligned}$$

The Shapiro-Shapiro conjecture says that as long as the lines in $\mathbb{P}^3$ are real, then this polynomial will have all real roots.

No no no, it's different. Let $AF - BE + CD$ be the Plücker embedding. Then write

$$\begin{pmatrix} A \\ B \\ C \\ D \\ E \\ F \end{pmatrix} = \begin{pmatrix} c_1 & d_1 \\ c_2 & d_2 \\ c_3 & d_3 \\ c_4 & d_4 \\ c_5 & d_5 \\ c_6 & d_6 \end{pmatrix} \begin{pmatrix} s \\ t \end{pmatrix}.$$

This parametrizes a line in $\mathbb{P}^5$, which will meet $Gr(2,4)$ in two points, which in $\mathbb{P}^3$ correspond to two (skew) lines. Now consider the 4×2 matrix

$$\begin{pmatrix} 1 & 0 \\ t & 1 \\ t^2 & 2t \\ t^3 & 3t^2 \end{pmatrix}$$

and put all 2×2 minors into a column matrix

$$\begin{pmatrix} 1 \\ 2t \\ 3t^2 \\ t^2 \\ 2t^3 \\ t^4 \end{pmatrix}.$$

Then for four different values $\alpha, \beta, \gamma, \delta$ of t, build a 6×4 matrix

$$M = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2\alpha & 2\beta & 2\gamma & 2\delta \\ 3\alpha^2 & 3\beta^2 & 3\gamma^2 & 3\delta^2 \\ \alpha^2 & \beta^2 & \gamma^2 & \delta^2 \\ 2\alpha^3 & 2\beta^3 & 2\gamma^3 & 2\delta^3 \\ \alpha^4 & \beta^4 & \gamma^4 & \delta^4 \end{pmatrix}$$

The nullspace of M is two-dimensional since $\alpha, \beta, \gamma, \delta$ are general. Choose a basis for the nullspace $\{\vec{v}, \vec{u}\}$. Then the parametrized line

$$\begin{pmatrix} A \\ B \\ C \\ D \\ E \\ F \end{pmatrix} = \begin{pmatrix} \vec{v} & \vec{u} \end{pmatrix} \begin{pmatrix} s \\ t \end{pmatrix}$$

meets $Gr(2,4)$ at all real points.