Let us have $S = \mathfrak{Z}(x^2+y^2+z^2-w^2) \subseteq \mathbb{P}^3_{\mathbb{C}}$ as our quadric surface. Let $N = (0,0,1,1) \in S$ and let $F = \mathfrak{Z}(z)$. Note that $L_1 = \mathfrak{Z}(x-iy, z-w), L_2 = \mathfrak{Z}(ix-y, z-w) \subseteq S$ are rulings at N.

Let us define a map $\pi : S \to F$ given by projection from N. Let $(a,b,c,d) \in S \setminus \{N\}$. Then the line connecting N and (a,b,c,d) can be parametrized by $(t,u) \in \mathbb{P}^1$ as

$$(ta, tb, (u-t)+tc, (u-t)+td).$$

This line intersects F when $(u-t)+tc = 0$, and so $u-(1-c)t = 0$, and so $(t,u) = (1, 1-c)$. Thus the line intersects F at $(a,b,0,d-c)$. Thus $\pi(a,b,c,d) = (a,b,0,d-c)$ for all $(a,b,c,d) \neq N$.

We run into a problem with this definition at N. We would have $\pi(N) = (0,0,0,1-1) = (0,0,0,0)$ which is not defined. To remedy this, we will need to blow up the quadric at N. By blowing up $\mathbb{P}^3$ at N, we are essentially replacing N with the set of all lines that contain N. This set of lines forms a plane. When we blow up S at N, we are only interested in such lines that are contained in $T_N(S) = \mathfrak{Z}(z-w)$, the tangent plane of S at N. This set of lines forms a line, which we may parametrize with $(t,u) \in \mathbb{P}^1$. For example, we will denote the line $\mathfrak{Z}(ux-ty, z-w)$ as (t,u). Note that the rulings L_1 and L_2 may be denoted $(i,1)$ and $(1,i)$, respectively.

Now that we have blown up S at N, our resulting surface will be denoted $\widetilde{S}$. It is not sufficient to speak of N, in $\widetilde{S}$, but rather $(N,(t,u))$. Note that $T_N(S) \cap F = \mathfrak{Z}(z,w) = \{(a,b,0,0)\}$. The line that connects N with $(a,b,0,0)$ is $\mathfrak{Z}(bx-ay, z-w)$, which we denote as (a,b). Thus when we develop a new map to extend π later, we can send $(N,(t,u))$ to $(t,u,0,0)$.

Let us return to the rulings L_1 and L_2. We can parametrize L_1 as (is, s, r, r) for some $(s,r) \in \mathbb{P}^1$. Note that for $s \neq 0$, $\pi(is,s,r,r) = (is,s,0,0) = (i,1,0,0)$. Therefore $\pi(L_1) = (i,1,0,0)$. Similarly, we have $\pi(L_2) = (1,i,0,0)$. This is not injective. To remedy this, we will need to blow up F at the two points $P_1 = (i,1,0,0)$ and $P_2 = (1,i,0,0)$. Blowing up F at P_1 and P_2 replaces each point with the set of lines that contain it. Thus in the blowup $\widetilde{F}$, we must write $(P_1,(t,u))$ instead of P_1 and $(P_2,(t,u))$ instead of P_2, where $(t,u) \in \mathbb{P}^1$.

The (t,u) in $(P_1,(t,u))$ will stand for the point $(t,0,0,u)$ on the x-axis $\mathfrak{Z}(y,z)$ that lays on the line through P_1. Likewise for $(P_2,(t,u))$.

Then in the extension of π, we can send (is,s,r,r) to $(P_1,(r,s))$ and send (s,is,r,r) to $(P_2,(r,s))$. Note that when $s = 0$, $(is,s,r,r) = (s,is,r,r) = N$ in S. In $\widetilde{S}$, we have $(is,s,r,r) = (N,(i,1))$ and $(s,is,r,r) = (N,(1,i))$ when $s = 0$. These points get sent to $(P_1,(1,0))$ and $(P_2,(1,0))$ respectively. The reason for this apparent backwards-ness is explained in the blue paragraph below.

Finally, we can formally describe the extension of π,

$$\widetilde{\pi} : \widetilde{S} \to \widetilde{F}$$

satisfying

$(N,(i,1)) \mapsto$ ~~$(P_1,(0,1))$~~ $(P_1,(1,0))$

$(N,(1,i)) \mapsto$ ~~$(P_2,(0,1))$~~ $(P_2,(1,0))$

$(N,(t,u)) \mapsto (t,u,0,0),\ (i,1) \neq (t,u) \neq (1,i)$

$(is,s,r,r) \mapsto$ ~~$(P_1,(s,r))$~~ $(P_1,(r,s)),\ (s,r) \neq (0,1)$

$(s,is,r,r) \mapsto$ ~~$(P_2,(s,r))$~~ $(P_2,(r,s)),\ (s,r) \neq (0,1)$

$(a,b,c,d) \mapsto (a,b,0,d-c)$, else.

We flip the r and s because the points $(P_1,(1,0))$ and $(P_2,(1,0))$ are collinear and the line connecting them is $\mathfrak{Z}(z,w)$, which is the image of $\{(N,(t,u))\}$. That is, $\mathfrak{Z}(z,w)$ is the image of the exceptional line of $\widetilde{S}$, and we represent this line with the pair $(1,0) \in \mathbb{P}^1$. This naming convention comes from the fact that $\mathfrak{Z}(z,w)$ intersects the x-axis $\mathfrak{Z}(y,z)$ at the point $(1,0,0,0)$. We then extend this necessity of the image of N-line to the images of the lines L_1 and L_2.

We claim that $\widetilde{\pi}$ is an isomorphism. We wish to show that blowing down the line in $\widetilde{F}$ that connects $(P_1,(1,0))$ and $(P_2,(1,0))$ (which is the lift of the line $\mathfrak{Z}(w)$ that connects P_1 and P_2 in F), creates a surface which is isomorphic to $\widetilde{S}$ blown down at the exceptional line $\widetilde{N} := \{(N,(t,u))\}$. We then wish to understand the blow-down of $\widetilde{F}$ as a $\mathbb{P}^1 \times \mathbb{P}^1$ and understand the graph of the real part of S as a subset of $\mathbb{P}^1 \times \mathbb{P}^1$.

Not sure why exactly $\widetilde{\pi}$ is an isomorphism. TBD.

We know that $\mathfrak{Z}(w)$ is the image of the exceptional line at N in $\widetilde{S}$, and so (given that $\widetilde{\pi}$ is indeed an isomorphism) blowing down the exceptional line in $\widetilde{S}$ and blowing down $\mathfrak{Z}(w)$ should maintain isomorphism.

Let $\widehat{F}$ be the blowdown of $\widetilde{F}$ at $\mathfrak{Z}(w)$. We claim that $\widehat{F} \cong \mathbb{P}^1 \times \mathbb{P}^1$. Let $(a,b,0,d) \in \widetilde{F} \setminus \mathfrak{Z}(w)$. Then we can represent $(a,b,0,d)$ with a line through P_1 and a line through P_2. Let us begin with the case that $(a,b,0,d)$ is on neither the exceptional lines at P_1 or P_2. Then in F, we can represent the line connecting P_1 and $(a,b,0,d)$ with a point along the x-axis collinear with $(a,b,0,d)$ and P_1. The line connecting $P_1 = (i,1,0,0)$ and $(a,b,0,d)$ is given by the polynomial

$$\det \begin{pmatrix} x & y & w \\ a & b & d \\ i & 1 & 0 \end{pmatrix} = x(-d) - y(-di) + w(a-ib).$$

This line intersects the x-axis $\mathfrak{Z}(y,z)$ at $(a-ib,0,0,d)$. Thus we can represent the line connecting P_1 and $(a,b,0,d)$ with the pair $(a-ib,d) \in \mathbb{P}^1$.

Similarly, we can represent the line connecting P_2 and $(a,b,0,d)$ with the pair $(a+ib,d) \in \mathbb{P}^1$. Thus we can represent $(a,b,0,d)$ in $\widehat{F}$ as $((a-ib,d),(a+ib,d))$.

Now let us pick a point $(P_1,(t,u))$ on the exceptional line at P_1. Then we know that $(P_1,(t,u))$ blows down to P_1 in F, whose connecting-line to P_2 is $\mathfrak{Z}(w)$, which intersects the x-axis at $(1,0)$. Thus we can represent $(P_1,(t,u))$ as $((t,u),(1,0))$. Similarly, we can represent $(P_2,(t,u))$ as $((1,0),(t,u))$. Thus we represent the entire line $\mathfrak{Z}(w)$ connecting $(P_1,(1,0))$ and $(P_2,(1,0))$ as $((1,0),(1,0))$. This is good, because we are blowing down this line. Note that

$\widetilde{N} \mapsto \mathfrak{Z}(w) \mapsto ((1,0),(1,0))$, and so in our map $\widehat{\pi} : S \to \widehat{F}$, we know $\widehat{\pi}(N) = ((1,0),(1,0))$. Also for $s \neq 0$, $\widehat{\pi}(is,s,r,r) = ((r,s),(1,0))$ and $\widehat{\pi}(s,is,r,r) = ((1,0),(r,s))$. For all other points, $\widehat{\pi}(a,b,c,d) = ((a-bi,d-c),(a+ib,d-c))$.

Now for a real point $(a,b,c,d) \in S$ (where a,b,c,d are all real), we can see the image $\widehat{\pi}(\mathbb{R}(S)) = \{((a-ib,d-c),(a+ib,d-c)) : a,b,c,d \in \mathbb{R}\} \cup \{((1,0),(1,0))\}$.

Also note that the total transform of the line L through P_1 and P_2 on F is $\widetilde{L} + E_1 + E_2$. Thus

$$1 = L^2 = (\widetilde{L}+E_1+E_2)^2 = \widetilde{L}^2+2\widetilde{L}E_1+2\widetilde{L}E_2+2E_1E_2+E_1^2+E_2^2 = \widetilde{L}^2+2+2-1-1 = \widetilde{L}+2.$$

Thus $\widetilde{L}^2 = -1$.

Now we will look at this through a more algebraic perspective. We will show that $\mathbb{P}^1 \times \mathbb{P}^1$ minus two lines is isomorphic to $\mathbb{P}^2$ minus one line by looking at the induced maps on their respective coordinate rings.

Note that $\mathbb{P}^1 \times \mathbb{P}^1 \cong Q = \mathfrak{V}(u_0u_3 - u_1u_2) \subseteq \mathbb{P}^3$. We can define a map

$$\mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow Q$$

given by

$$((a:b),(c:d)) \longmapsto (ac:ad:bc:bd)$$

and an inverse map

$$Q \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$$

given by

$$(u_0:u_1:u_2:u_3) \longmapsto ((u_0+u_1:u_2+u_3),(u_0+u_2:u_1+u_3)).$$

Let $U = \mathbb{P}^2 \setminus \mathfrak{V}(z) = \mathfrak{D}_+(z)$. Then $\mathcal{O}_{\mathbb{P}^2}(U) = k[x,y,z,z^{-1}]$. We can choose $(1:0:0)$ and $(0:1:0) \in \mathfrak{V}(z)$ as our two blowup points. Then any point on U can be uniquely determined by the slopes of the lines connecting it with the two points at infinity (the point's width and height). Then we can define a birational map

$$\mathbb{P}^2 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow Q$$

given by

$$(a:b:c) \longmapsto ((a:c),(b:c)) \longmapsto (ab:ac:bc:c^2).$$

Then we can find an open subset $V \subseteq Q$ and a map

$$\mathcal{O}_Q(V) = k[u_0,u_1,u_2,u_3]/(u_0u_3-u_1u_2)_V \longrightarrow k[x,y,z,z^{-1}]$$

satisfying

$$\begin{aligned} u_0 &\longmapsto xy, \\ u_1 &\longmapsto xz, \\ u_2 &\longmapsto yz, \\ u_3 &\longmapsto z^2. \end{aligned}$$

Thus $V = Q \setminus \mathfrak{V}(u_3)$ so that we can invert u_3. Note that $Q \cap \mathfrak{V}(u_3) = \mathfrak{V}(u_1, u_3) \cup \mathfrak{V}(u_2, u_3)$, a union of two lines! Thus

$$\mathcal{O}_Q(V) = k[u_0, u_1, u_2, u_3, u_3^{-1}]/(u_0u_3 - u_1u_2) \cong k[u_1, u_2, u_3, u_3^{-1}]$$

which is isomorphic to $k[x, y, z, z^{-1}] = \mathcal{O}_{\mathbb{P}^2}(\mathfrak{D}_+(z))$. Note that the map given above is injective but not surjective; but its image is $k[xz, yz, z^2, z^{-2}]$, which is isomorphic to $k[x, y, z, z^{-1}]$!!

Thus the quadric minus two lines ($Q \cap V(u_3) = \mathfrak{V}(u_1, u_3) \cup \mathfrak{V}(u_2, u_3)$) is isomorphic to $\mathbb{P}^2$ minus one line ($\mathfrak{V}(z)$). Thus the quadric is rational.