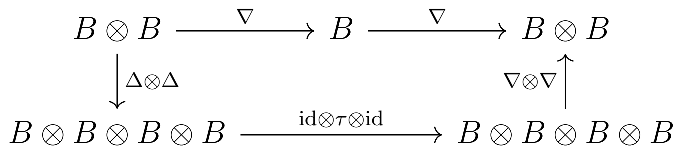
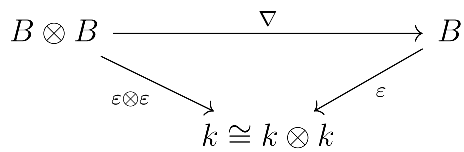
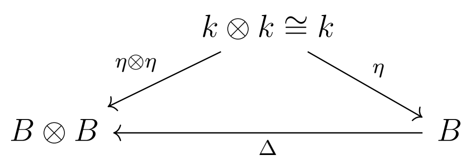
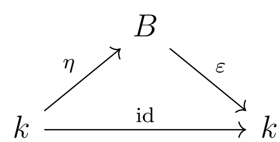
Definition 1. A **bialgebra** $(B, \nabla, \eta, \Delta, \varepsilon)$ comprises a k-vector space B together with a multiplication map $\nabla : B \otimes B \to B$, a unit $\eta : k \to B$, a comultiplication map $\Delta : B \to B \otimes B$ and a counit $\varepsilon : B \to k$, such that the following four diagrams commute.



(where $\tau : B \otimes B \to B \otimes B$ is a linear function defined by $\tau(x \otimes y) = y \otimes x$)





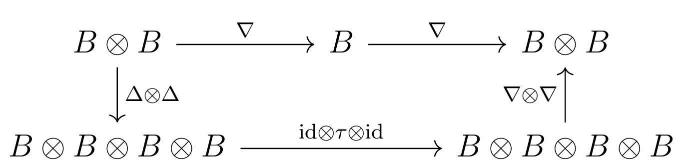


Example 1. Let $B = k[x]$ equipped with the following maps.

$$\nabla(x^n \otimes x^m) = x^{n+m}$$
$$\eta(1) = 1$$
$$\Delta(x^k) = \sum_{i=0}^{k} \binom{k}{i} x^i \otimes x^{k-i}$$
$$\varepsilon(x^k) = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases}$$

Then $(B, \nabla, \eta, \Delta, \varepsilon)$ is a k-bialgebra.

Proof. First let us look at the diagram



and consider the top route. We start with $x \otimes x \in B \otimes B$. Then $\nabla(x \otimes x) = x^2$ and $\Delta(x^2) = x^2 \otimes 1 + 2x \otimes x + 1 \otimes x^2$.

Now

$$\begin{aligned}\Delta \otimes \Delta(x \otimes x) &= \Delta(x) \otimes \Delta(x) = (x \otimes 1 + 1 \otimes x) \otimes (x \otimes 1 + 1 \otimes x) \\ &= x \otimes 1 \otimes x \otimes 1 + x \otimes 1 \otimes 1 \otimes x + 1 \otimes x \otimes x \otimes 1 + 1 \otimes x \otimes 1 \otimes x.\end{aligned}$$

Applying $\mathrm{id} \otimes \tau \otimes \mathrm{id}$ to the above sum gives us

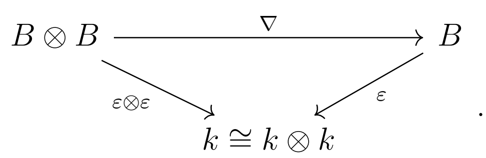
$$x \otimes x \otimes 1 \otimes 1 + x \otimes 1 \otimes 1 \otimes x + 1 \otimes x \otimes x \otimes 1 + 1 \otimes 1 \otimes x \otimes x$$

(just swapping the middle two of each term). Now applying $\nabla \otimes \nabla$ to this sum yields

$$x^2 \otimes 1 + x \otimes x + x \otimes x + 1 \otimes x^2 = x^2 \otimes 1 + 2x \otimes x + 1 \otimes x^2.$$

So we can see that the diagram commutes for the element $x \otimes x$! The same applies for a general $x^n \otimes x^m$, but the polynomials get much messier!

Now let us consider the diagram



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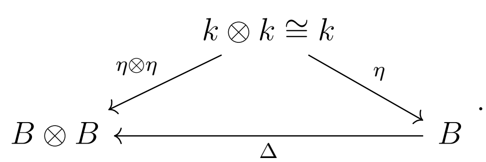
Let $x^n \otimes x^m \in B \otimes B$. Then $\nabla(x^n \otimes x^m) = x^{n+m}$. Then $\varepsilon(x^{n+m}) = 0$ if $n + m \geq 0$ and 1 if $n + m = 0$. The latter can only be the case if $n = m = 0$, in which case $(\varepsilon \otimes \varepsilon)(x^n \otimes x^m) = \varepsilon(x^0) \otimes \varepsilon(x^0) = 1 \otimes 1 = 1 \in k$. If the former is the case, then at least one of n or m is > 0 (WLOG $n > 0$). In this case $(\varepsilon \otimes \varepsilon)(x^n \otimes x^m) = 0 \otimes \varepsilon(x^m) = 0 \in k$. So the diagram commutes.

Now let us look at the diagram



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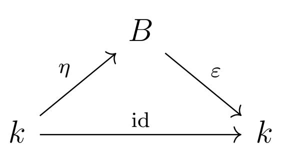
Let $a \in k$. Then $a = a \otimes 1 \in k \otimes k$. We have $\eta(a) = a$, and

$$\Delta(a) = \Delta(ax^0) = a\Delta(x^0) = a\binom{0}{0}x^0 \otimes x^0 = a(1 \otimes 1) = a \otimes 1.$$

Furthermore, $(\eta \otimes \eta)(a \otimes 1) = a \otimes 1$. So the diagrams commute.

Finally, let us consider the diagram



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Let $a \in k$. Then $\eta(a) = a$ and $\varepsilon(a) = a$. So the diagram commutes. $\square$

Definition 2. A bialgebra's comultiplication is **coassociative** if $(\mathrm{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}) \circ \Delta$ and its map ε is a **counit** if $(\mathrm{id} \otimes \varepsilon) \circ \Delta = \mathrm{id} = (\varepsilon \otimes \mathrm{id}) \circ \Delta$. In other words, the following diagrams commute.

$$\begin{array}{ccc} B & \xrightarrow{\Delta} & B \otimes B \\ \big\downarrow{\scriptstyle \Delta} & & \big\downarrow{\scriptstyle \mathrm{id}\otimes\Delta} \\ B \otimes B & \xrightarrow{\Delta\otimes\mathrm{id}} & B \otimes B \otimes B \end{array}$$

$$\begin{array}{ccc} B & \xrightarrow{\Delta} & B \otimes B \\ \big\downarrow{\scriptstyle \Delta} & \searrow^{\mathrm{id}} & \big\downarrow{\scriptstyle \mathrm{id}\otimes\varepsilon} \\ B \otimes B & \xrightarrow{\varepsilon\otimes\mathrm{id}} & k \otimes B = B = B \otimes k \end{array}$$

Example 2. The comultiplication and counit from the above example $(B = k[x], \nabla, \eta, \Delta, \varepsilon)$ satisfy these commutative diagrams.

Proof. First let $x \in B$. Then $\Delta(x) = x \otimes 1 + 1 \otimes x$ and

$$(\mathrm{id} \otimes \Delta)(x \otimes 1 + 1 \otimes x) = x \otimes 1 \otimes 1 + 1 \otimes (x \otimes 1 + 1 \otimes x) = x \otimes 1 \otimes 1 + 1 \otimes x \otimes 1 + 1 \otimes 1 \otimes x.$$

It is straightforward to see that we get the same thing going the other way around the diagram. Therefore the first diagram commutes for $x \in B$! (Again, higher degree polynomials get gross.)

Now for the second diagram. Let $x \in B$. Then $\Delta(x) = x \otimes 1 + 1 \otimes x$. Then

$$(\mathrm{id} \otimes \varepsilon)(x \otimes 1 + 1 \otimes x) = x \otimes 1 + 1 \otimes 0 = x \otimes 1 = x = \mathrm{id}(x).$$

It is straightforward to see the other route matches. Therefore the second diagram commutes for $x \in B$! Higher degrees blah blah blah.... $\square$

Definition 3. A bialgebra is a **Frobenius algebra** if the following diagrams commute.

$$\begin{array}{ccc} B \otimes B & \xrightarrow{\Delta\otimes\mathrm{id}} & B \otimes B \otimes B \\ \big\downarrow{\scriptstyle \nabla} & & \big\downarrow{\scriptstyle \mathrm{id}\otimes\nabla} \\ B & \xrightarrow{\Delta} & B \otimes B \end{array}$$

$$\begin{array}{ccc} B \otimes B & \xrightarrow{\mathrm{id}\otimes\Delta} & B \otimes B \otimes B \\ \big\downarrow{\scriptstyle \nabla} & & \big\downarrow{\scriptstyle \nabla\otimes\mathrm{id}} \\ B & \xrightarrow{\Delta} & B \otimes B \end{array}$$

A Frobenius algebra is additionally called **special** (or **isometric**) if $\nabla \circ \Delta = \mathrm{id}$.

Remark 1. The above example $(B = k[x], \nabla, \eta, \Delta, \varepsilon)$ is not a Frobenius algebra, and neither does it satisfy the special condition.

Indeed, considering the first diagram, let $x \otimes x \in B \otimes B$. Then

$$(\Delta \otimes \mathrm{id})(x \otimes x) = (x \otimes 1 + 1 \otimes x) \otimes x = x \otimes 1 \otimes x + 1 \otimes x \otimes x.$$

Then applying $\mathrm{id} \otimes \nabla$ to this sum yields

$$x \otimes x + 1 \otimes x^2.$$

Now going back and considering $\Delta \circ \nabla(x \otimes x)$, we get $\nabla(x \otimes x) = x^2$ and

$$\Delta(x^2) = x^2 \otimes 1 + 2x \otimes x + 1 \otimes x^2 \neq x \otimes x + 1 \otimes x^2,$$

so the diagram does not commute! The same thing happens with the bottom diagram, so neither of them commute.

It should be noted that $\Delta \circ \nabla \neq (\mathrm{id} \otimes \nabla) \circ (\Delta \otimes \mathrm{id}) + (\nabla \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \Delta)$ in general (but the equality holds for $x \otimes x$). As far as I'm aware, there is no name for this property.

Observe: consider $x^n \otimes x^m$. Then $\nabla(x^n \otimes x^m) = x^{n+m}$. Then

$$\Delta(x^{n+m}) = \sum_{i=0}^{n+m} \binom{n+m}{i} x^i \otimes x^{n+m-i}.$$

Furthermore,

$$\Delta(x^n \otimes \mathrm{id})(x^n \otimes x^m) = \sum_{i=0}^{n} \binom{n}{i} (x^i \otimes x^{n-i} \otimes x^m).$$

Then applying $(\mathrm{id} \otimes \nabla)$, we get

$$\sum_{i=0}^{n} \binom{n}{i} x^i \otimes x^{n+m-i}.$$

Going the other way we apply $(\nabla \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \Delta)$ to $x^n \otimes x^m$ and we get

$$\sum_{j=0}^{m} \binom{m}{j} x^{n+j} \otimes x^{m-j}.$$

Now consider $(\nabla \circ \Delta) : B \to B$. Consider $x \in B$. Then $\Delta(x) = x \otimes 1 + 1 \otimes x$. Then $\nabla(x \otimes 1 + 1 \otimes x) = x + x = 2x \neq x$. Therefore this bialgebra is not special. In fact, $(\nabla \otimes \Delta)(x^n) = 2^n x^n$. This is because $\sum_{i=0}^{n} \binom{n}{i} = 2^n$.

Also $B \otimes B$ is the same thing as the set of functions from $\mathbb{N} \times \mathbb{N}$ to k where all but finitely many of each input's output is 0, under

$$\left(\sum_{(i,j) \in \mathbb{N} \times \mathbb{N}} \alpha_{i,j} x^i \otimes x^j \right) (i,j) = \alpha_{i,j}.$$