The following is notes on Dolgachev's paper on the Chilean configuration of points. In chapter 6, they discuss the blowup $Y$ of $\mathbb{P}^2$ at eight of the nine points of the Chilean configuration $P_1, \dots, P_8$ and the map $|-2K_Y| : Y \to \mathbb{P}^3$ given by $p \mapsto (f(p)^2, f(p)g(p), g(p)^2, \varphi(p))$ (the generators of $|-2K_Y|$) where $f$ and $g$ are generators of $|-K_Y|$. We know there are four generators of $|-2K_Y|$ because $\binom{6+2}{2} - 8\binom{2+1}{2} = 4$.

Let $P_9$ be the ninth point of the Chilean configuration and let $Q \in \mathbb{P}^2$ be such that $P_1, \dots, P_8, Q$ are in Cayley-Bacharach position. The antibicanonical map (henceforth denoted $\beta$) induces the *double plane model* of $Y$. The image of $\beta$ is a cone $V(xz - y^2)$ with a singularity at $(0, 0, 0, 1)$. We will show that this map is degree 2, and that $\beta(p) = \beta(q)$ if and only if $p = -q \mod Q$ (in other words, $p$ and $q$ map to each other under the Bertini involution).

First we will show that if $\beta(p) = \beta(q)$, $p$ and $q$ are on the same fiber. Given $\beta(p) = \beta(q)$, we have $(f(p)^2, f(p)g(p), g(p)^2, \varphi(p)) = (f(q)^2, f(q)g(q), g(q)^2, \varphi(q)) \in \mathbb{P}^3$, or

$$\operatorname{rank}\begin{pmatrix} f(p)^2 & f(p)g(p) & g(p)^2 & \varphi(p) \\ f(q)^2 & f(q)g(q) & g(q)^2 & \varphi(q) \end{pmatrix} = 1.$$

We wish to show that there is a $(\lambda, \mu) \in \mathbb{P}^1$ such that $\lambda f(p) + \mu g(p) = \lambda f(q) + \mu g(q) = 0$. First suppose $\varphi(p), \varphi(q) \neq 0$. Then

$$\left(\frac{f(p)^2}{\varphi(p)}, \frac{f(p)g(p)}{\varphi(p)}, \frac{g(p)^2}{\varphi(p)}\right) = \left(\frac{f(q)^2}{\varphi(q)}, \frac{f(q)g(q)}{\varphi(q)}, \frac{g(q)^2}{\varphi(q)}\right) \in \mathbb{A}^3.$$

Now suppose $p$ is on the $(\lambda, \mu)$-fiber; in other words, $\lambda f(p) + \mu g(p) = 0$. Then

$$(\lambda f(p) + \mu g(p))^2 = \lambda^2 f(p)^2 + 2\lambda\mu f(p)g(p) + \mu^2 g(p) = 0$$

and so

$$\lambda^2 \frac{f(p)^2}{\varphi(p)} + 2\lambda\mu \frac{f(p)g(p)}{\varphi(p)} + \mu^2 \frac{g(p)^2}{\varphi(p)} = \lambda^2 \frac{f(q)^2}{\varphi(q)} + 2\lambda\mu \frac{f(q)g(q)}{\varphi(q)} + \mu^2 \frac{g(q)^2}{\varphi(q)} = 0$$

and so $(\lambda f(q) + \mu g(q))^2 = 0$ so $q$ is on the $(\lambda, \mu)$-fiber along with $p$.

Next suppose $\beta(p) = \beta(q)$ and $\varphi(p) = \varphi(q) = 0$. Then the same argument as above holds, just instead of dividing my $\varphi(p)$ or $\varphi(q)$ just divide by some nonzero scalar $s = f(q)^2/f(p)^2$.

Next we will show that $\beta(p) = \beta(q)$ implies $p = -q \mod Q$. That is, $p + q \sim 2Q$. First let us consider the case $p, q$ are on the $(1, 0)$-fiber, so $f(p) = f(q) = 0$. Let us analyze the regular function $g^2/\varphi$. Note that $\operatorname{div}(g^2/\varphi) = 2P_1 + \dots + 2P_8 + 2Q - 2P_1 - \dots - 2P_8 - A_1 - A_2 = 2Q - A_1 - A_2$ for some $A_1, A_2 \in V(f) \cap V(\varphi)$. Our goal will be to take linear combinations of $g^2$ and $\varphi$ to get $A_1$ and $A_2$ to line up with $p$ and $q$.

We know that $\varphi$ and $f$ intersect at the 18 points $2P_1, \dots, 2P_8, A_1, A_2$. Let $(a, b)$ be such that $ag^2(p) + b\varphi(p) = 0$ and $f(p) = 0$. We then want to show that $ag^2(q) + b\varphi(q) = 0$ and $f(q) = 0$ given that $\beta(p) = \beta(q)$. First, we show that $f(q) = 0$. We know that

$$\operatorname{rank}\begin{pmatrix} 0 & 0 & g(p)^2 & \varphi(p) \\ f(q)^2 & f(q)g(q) & g(q)^2 & \varphi(p) \end{pmatrix} = 1$$

so we must have $f(q) = 0$. Furthermore, we know that

$$\operatorname{rank}\begin{pmatrix} 0 & 0 & g(p)^2 & \varphi(p) \\ 0 & 0 & g(q)^2 & \varphi(q) \end{pmatrix} = 1$$

since $\beta(p) = \beta(q)$. Therefore $bg(p)^2\varphi(q) = bg(q)^2\varphi(p)$. Since $ag(p)^2 + b\varphi(p) = 0$, we know that $bg(p)^2\varphi(q) = -ag(q)^2g(p)^2$. Assuming $g(p) \neq 0$ (otherwise $(a, b) = (1, 0)$) we can cancel $g(p)^2$ and thus

$$b\varphi(q) = -ag(q)^2,$$

and so $ag(q)^2 + b\varphi(q) = 0$. Therefore $\operatorname{div}_f(g^2/(ag^2 + b\varphi)) = 2Q - p - q$. Thus $2Q \sim p + q$.

Other direction written now: assume that $p+q \sim 2Q$, then $\beta(p) = \beta(q)$. Since $p+q \sim 2Q$, we know that $p + q + 2P_1 + \cdots + 2P_8 \sim 2Q + 2P_1 + \cdots + 2P_8$ and so there is a $(\lambda, \mu) \in \mathbb{P}^1$ such that $\lambda g(p)^2 + \mu\varphi(p) = \lambda g(q)^2 + \mu\varphi(q) = 0$. Then

$$\frac{\varphi(p)}{g(p)^2} = -\frac{\lambda}{\mu} = \frac{\varphi(q)}{g(q)^2}$$

and so $\beta(p) = \beta(q) = (0, 0, -\mu, \lambda)$. The reverse direction is just the proof in reverse.

The idea is similar if $p, q \in V(\lambda f + \mu g)$, just more complicated. Maybe I'll write it down one day.

Let $\lambda f(p) + \mu g(p) = \lambda f(q) + \mu g(q) = 0$ and $\beta(p) = \beta(q)$. Now let $af(p)^2 + bf(p)g(p) + cg(p)^2 + d\varphi(p) = 0$ for some $(a, b, c, d) \in \mathbb{P}^3$. Then we want to show that $af(q)^2 + bf(q)g(q) + cg(q)^2 + d\varphi(q) = 0$. We know that

$$\operatorname{rank}\begin{pmatrix} f(p)^2 & f(p)g(p) & g(p)^2 & \varphi(p) \\ f(q)^2 & f(q)g(q) & g(q)^2 & \varphi(q) \end{pmatrix} = 1$$

and that the top row satisfies the equation $ax_0 + bx_1 + cx_2 + dx_3 = 0$. Therefore the bottom row must satisfy the same plane equation, and so

$$af(q)^2 + bf(q)g(q) + cg(q)^2 + d\varphi(q) = 0.$$

Thus $\operatorname{div}_{\lambda f+\mu g}(g^2/(af^2 + bfg + cg^2 + d\varphi)) = 2Q - p - q$, so $p + q \sim 2Q$.

I also have a more abstract argument than Moody's for why the image of each exceptional divisor is a curve that has a triple point.

$$e_1 \mapsto 4\ell - 3e_1 - 2e_2 - \cdots - 2e_8$$

under the Bertini involution on the blowup of four points.

We want to know how many times on the pencil $\langle f, g\rangle$ does $\tau P = \tau Q$, where $P$ is the point in $\mathbb{P}^2$ that is the blowdown of $e_1$ and $Q$ is the ninth base point of $|-K_Y|$. When $\tau P = \tau Q$, then $P = -P \mod Q$ and so $\beta(e_1)$ will intersect $e_1$ at the corresponding point.

When $\tau P = \tau Q$, then $2P \sim 2Q$ as Weil divisors on the fibre $\lambda f + \mu g$. Then

$$\underbrace{4P + 2P_2 + \cdots + 2P_8}_{\Delta} \sim \underbrace{2Q + 2P + 2P_2 + \cdots + 2P_8}_{(af+bg)^2 \neq (\lambda f + \mu g)^2},$$

where $\Delta$ is the unique sextic curve that has seven double points at $P_2, \ldots, P_8$ and a triple point at $P$. Notice that

$$\binom{6+2}{2} - 7\binom{2+1}{2} - \binom{3+1}{2} = 1,$$

so such a sextic does indeed exist and is unique. In order for $\lambda f + \mu g$ to intersect $\Delta$ with multiplicity 4 at $P$, $\lambda f + \mu g$ must have a tangent direction at $P$ equal to one of the three tangent directions of $\Delta$ at $P$. Therefore there are three possibilities for $(\lambda, \mu) \in \mathbb{P}^1$, and so there are three fibres whereupon $P = -P \mod Q$.

Similarly, $e_2$ has a point whose inverse mod $Q$ is on $e_1$ when $P_2 = -P \mod Q$ on a fibre. Then $\tau Q = \overline{PP_2}$ and so $2Q \sim P + P_2$. Then

$$3P + 3P_2 + 2P_3 + \cdots + 2P_8 \sim 2Q + 2P + 2P_2 + \cdots + 2P_8.$$

Now we must answer "for how many $(\lambda, \mu) \in \mathbb{P}^1$ does there exist a sextic whose intersection product with $\lambda f + \mu g$ is $3P + 3P_2 + 2P_3 + \cdots + 2P_8$?" We can design a sextic with six double points at $P_3, \ldots, P_8$ and have

$$28 - 6 * 3 = 10 \text{ conditions to spare.}$$

If we just use $\Delta$ again, we can see that there will be 2 $(\lambda, \mu)$-points. The tangent line of $\lambda f + \mu g$ will intersect $\Delta$ at $P_2$ with multiplicity 3 twice: once for each time the tangent line of $\lambda f + \mu g$ at $P_2$ equals one of the two tangent lines of $\Delta$ at $P_2$. Therefore there are two fibres whereupon $P = -P_2 \mod Q$.

I want to prove that a base point of general pencil will be flex on three fibres (and every other base point is equal to $\tau P$ on one fibre). For the fibre $\lambda f + \mu g$, $P$ will be flex if $3P \sim \ell$. So for example $3P \sim 2Q + \tau Q$, or $\tau Q = -\beta(P) \mod P$.

There exists a sextic curve that whose intersection product with $\lambda f + \mu g$ is $P + 2P_2 + \cdots + 2P_8 + \ell$ where $\ell$ is the class of three colinnear points (in fact there is a $\mathbb{P}^2$ of sextics that have seven double points and four normal points). Then

$$3P \sim \ell$$

is equivalent to

$$4P + 2P_2 + \cdots + 2P_8 \sim P + 2P_2 + \cdots 2P_8 + \ell.$$

There exists a unique sextic curve $\Delta$ that has a triple point at $P$ and double points at $P_2, \ldots, P_8$. Then

$$\Delta.(\lambda f + \mu g) = 4P + 2P_2 + \cdots + 2P_8$$

only when $\lambda f + \mu g$ is tangent to one of the slopes of $\Delta$ at $P$. Since it is a triple point, this happens three times.