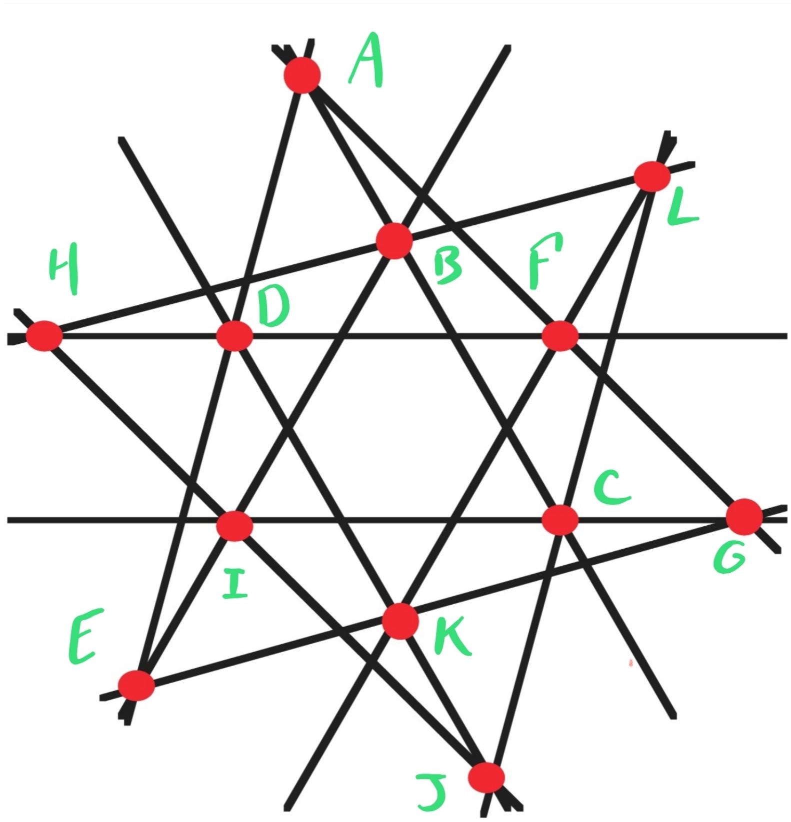
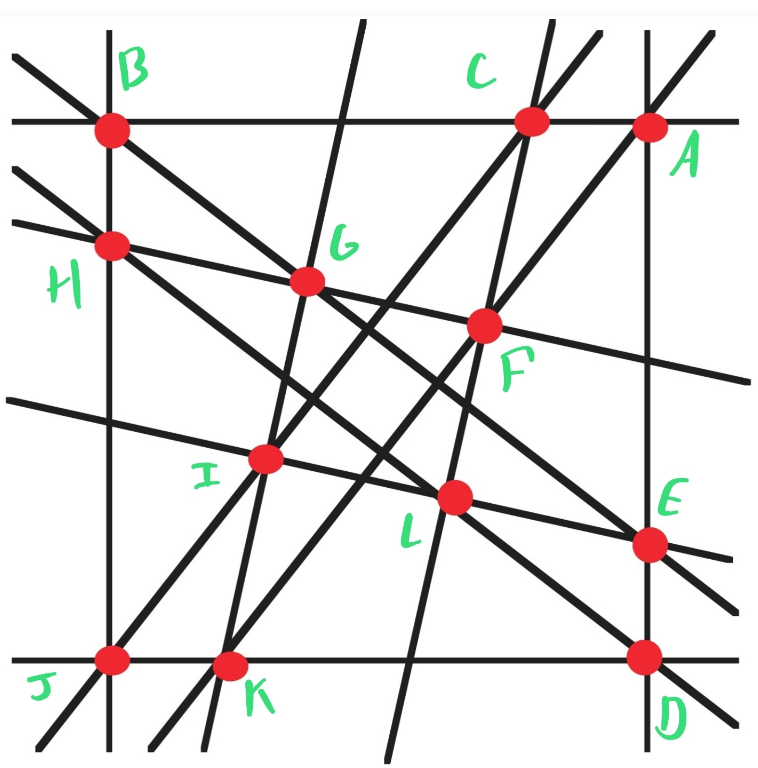
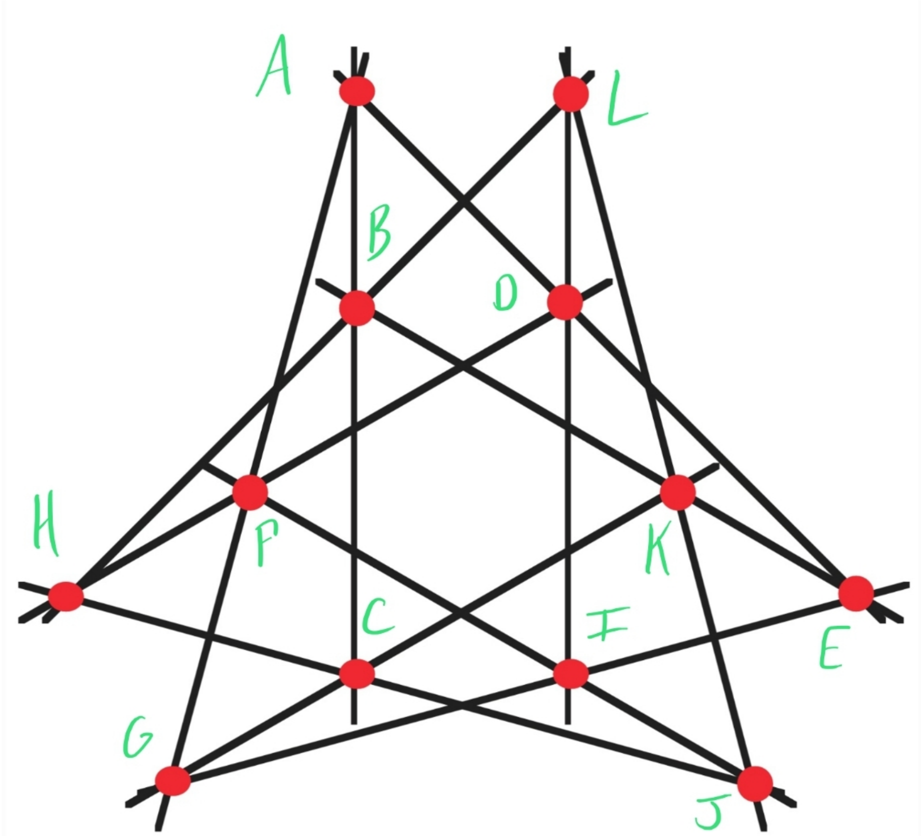
The following comes from Problem 2 on page 25 of Grünbaum's book on configurations of points and lines. Consider the three (12_3) configurations shown below.







Let us define a **deficiency simplex** as a simplex whose vertices are points of a configuration such that:

- No two vertices in the simplex correspond to points sharing a line in the configuration,
- Every point on the configuration shares a line with at least one of the points in the simplex.

The union of deficiency simplicies of a configuration is the **deficiency complex** of the configuration. For example, the first configuration has a deficiency complex shown here. Topologically, it is an annulus with three interior 1-simplicies and three exterior 1-simplicies, giving a total rank of the first homology group 7.

We can confirm this computationally using `Macaulay2`:

```
loadPackage "SimplicialComplexes"

R=QQ[a,b,c,d,e,f,g,h,i]

L={a*i*l,a*h*k,a*l*k,a*j,b*g*j,b*d*g,b*f*j,b*k,c*e*f,c*h*k,c*h*e,c*d,
d*i*l,d*g*i,e*f*j,e*i,f*l,g*h}

OCT = simplicialComplex L
```

```
fVector OCT

CHAIN=chainComplex OCT

rank HH_2 CHAIN

rank HH_1 CHAIN

rank HH_0 CHAIN
```

We see that the rank `HH_1` is 7, and the rest are 0.

For the second configuration, we see that the deficiency complex comprises the 2-simplicies $\{A, H, I\}$, $\{B, L, K\}$, $\{C, D, G\}$, and $\{E, F, J\}$, and the 3-simplicies $\{A, G, J, L\}$, $\{C, E, H, K\}$, and $\{B, D, F, I\}$. Putting these in Macaulay2, we get

```
loadPackage "SimplicialComplexes"

R=QQ[a,b,c,d,e,f,g,h,i]

L={a*h*i,a*g*j*l,c*d*g,c*e*h*k,b*f*i*d,b*l*k,e*j*f}

OCT = simplicialComplex L

fVector OCT

CHAIN=chainComplex OCT

rank HH_3 CHAIN

rank HH_2 CHAIN

rank HH_1 CHAIN

rank HH_0 CHAIN
```

and we get a `rank HH_1` of 6 and the rest 0. Topologically, this simplicial complex is like three disjoint D^3's with four disjoint D^2's, each D^2 meeting each D^3 at one point.

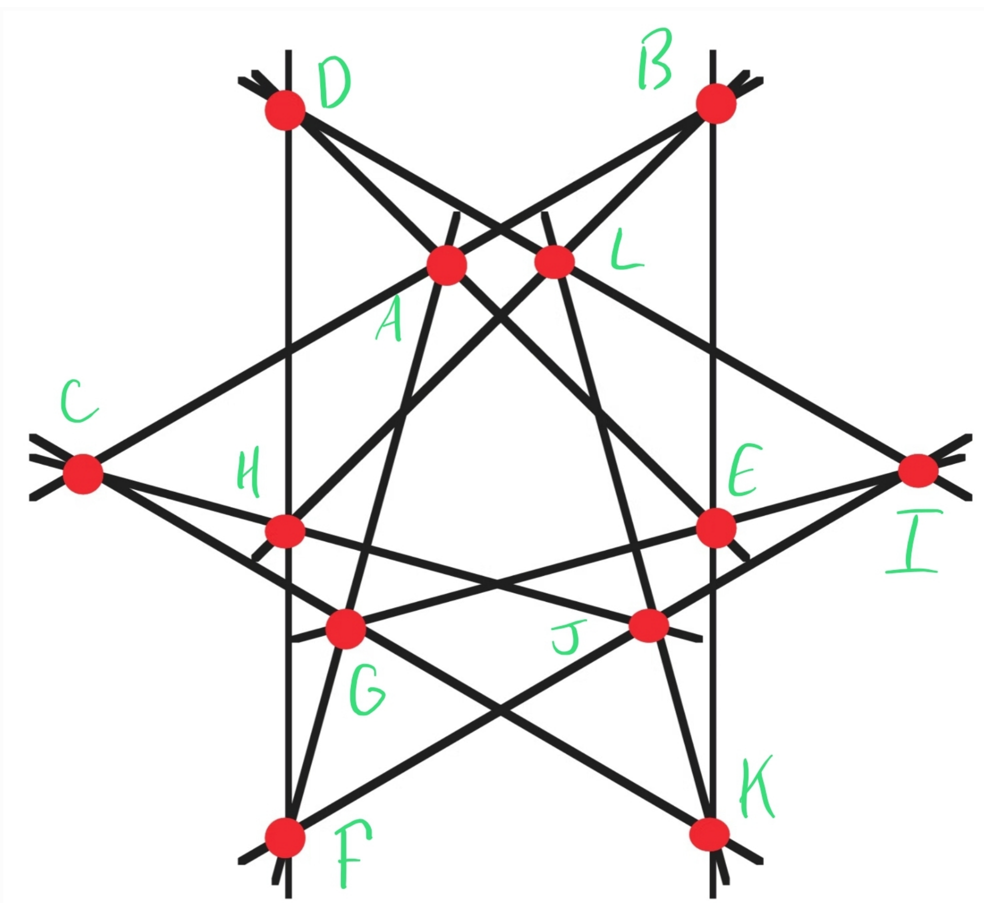
Finally, we have the third configuration with a deficiency complex comprising the 1-simplicies $\{A, J\}$, $\{A, L\}$, $\{B, F\}$, $\{B, I\}$, $\{C, D\}$, $\{C, I\}$, $\{D, K\}$, $\{E, H\}$ $\{E, J\}$, $\{F, K\}$, $\{G, H\}$, and $\{G, L\}$, and the 3-simplicies $\{A, H, I, K\}$, $\{B, D, G, J\}$, and $\{C, E, F, L\}$. Putting this into Macaulay2, we get

```
loadPackage "SimplicialComplexes"

R=QQ[a,b,c,d,e,f,g,h,i]
```

```
L={a*h*i*k,a*j,a*l,b*d*g*j,b*f,b*i,c*d,c*f*l*e,c*i,d*k,e*j,f*k,g*h,g*l,e*h}

OCT = simplicialComplex L

fVector OCT

CHAIN=chainComplex OCT

rank HH_3 CHAIN

rank HH_2 CHAIN

rank HH_1 CHAIN

rank HH_0 CHAIN
```

and a `rank HH_1` of 10 and the rest 0. Topologically, this simplicial complex is like three disjoint D^3's each connected to the other two by four line segments.

Finally, Grünbaum's book also asks about a fourth (12$_3$) configuration, shown below.



This is be seen to be isomorphic with Configuration 3 by observing the choice of labeling as the isomorphism.