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Part 1: Functions

Definition 1. A **function** from the set A to the set B takes as inputs the elements of A and assigns each input to one element of B. The sentence "f is a function from A to B" is written as "$f : A \to B$." Here A is the **domain** of f and B is the **codomain**.

IMPORTANT: Two functions $f : A \to B$ and $g : X \to Y$ are the same if and only if $A = X$ and $B = Y$ and $f(a) = g(a)$ for all $a \in A$. Then we write $f = g$.

The functions $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = x^2$, $g : \mathbb{R} \to [0, \infty)$ defined as $g(x) = x^2$, and $h : [0, \infty) \to [0, \infty)$ defined as $h(x) = x^2$ are three different functions.

Example 1. Let P be the set of people and let D be the set of days. Then there is a function $f : P \to D$ that receives the input of a person and outputs that person's birthday.

Example 2. Let P be the set of people and let J be the set of days in June. Then the relation $g : P \to J$ that receives the input of a person and outputs that person's birthday is **not** a function, because not everyone's birthday is in June, so those inputs have no output.

Example 3. Let D be the set of days and let P be the set of people. Then the relation $h : D \to P$ that receives the input of a day and outputs the people having that day as a birthday is **not** a function because more than one person is born on a given day.

Definition 2. Let $f : A \to B$ be a function. The **image** (aka **range**) of f is the set $\operatorname{img}(f) = \{b \in B : b = f(a) \text{ for some } a \in A\}$. That is, the image of f is the set of all outputs.

Definition 3. Let $f : A \to B$ be a function. Then f is **injective** (aka **one-to-one**) if $f(a) = f(a')$ implies $a = a'$. (Intuitively, this means that no two inputs have the same output.)

Definition 4. Let $f : A \to B$ be a function. Then f is **surjective** (aka **onto**) if $\operatorname{img}(f) = B$. (Intuitively, if every element of the codomain is the output of some input from the domain.)

Definition 5. Let $f : A \to B$ be a function. Then f is **bijective** (aka **invertible**) if f is both injective and surjective.

Definition 6. Let $f : A \to B$ be a function. Then an **inverse** of f is a function $g : B \to A$ such that $g(f(a)) = a$ and $f(g(b)) = b$ for all $a \in A$ and all $b \in B$.

Inverses are unique.

Proof. Let $f : A \to B$, and let $g : B \to A$ and $h : B \to A$ both be inverses of f, and we will prove that $g = h$. Let $b \in B$. Then $g(b) = h(f(g(b))) = h(b)$. So $g = h$. □

Since inverses are unique, it makes sense to speak of *the* inverse of a function f, which we can write as f^{-1}.

Bijective functions have inverses.

Proof. Let $f : A \to B$ be bijective. Then for all outputs $b \in B$ there is exactly one element $a \in A$ such that $f(a) = b$. Then we can construct a function $g : B \to A$ such that $g(b)$ is this unique element of A. This is an inverse of f because $g(f(a)) = a$ by definition and $f(g(b)) = b$ by definition. □

Non-injective functions are not invertible.

If $f : A \to B$ is not injective, then $a, a' \in A$ where $a' \neq a$ and $f(a) = f(a') = b$. Then the supposed "inverse" will have more than one output for the input b, so cannot be a function.

Non-surjective functions are not invertible

If $f : A \to B$ is not surjective, then there is some $b \in B$ such that $f(a) \neq b$ for all $a \in A$. Then the supposed "inverse" will not provide an output for b, so cannot be a function.

Thus **only bijective functions have inverses** (hence the name 'invertible').

The existence of a bijection between two sets A and B indicates that A and B are the "same" in a sense, even if they are not *literally* the same. Later on we will see this "sameness" has a more specific name: *isomorphic!*

Example 4. There is no bijection between $A = \{1, 2\}$ and $B = \{1, 2, 3\}$. Any function $f : A \to B$ will fail to be surjective, and any function $g : B \to A$ will fail to be injective. A is smaller than B.

Example 5. The sets $A = \{1, 2, 3\}$ and $B = \{11, 12, 13\}$ have a bijection $f : A \to B$ satisfying $f(x) = x + 10$. A and B have the same size (aka "cardinality").

Part 2: Linear Transformations

Definition 7. Let V and U be vector spaces. A **linear transformation** from V to U is a function $T : V \to U$ satisfying the two conditions:

- $T(v + v') = T(v) + T(v')$ for all $v, v' \in V$,
- $T(sv) = sT(v)$ for all $s \in \mathbb{R}$ and $v \in V$.

Example 6. The function $T : \mathbb{R}^2 \to \mathbb{R}^3$ satisfying $T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} 2a + 3b \\ 5a - 7b \\ 11a + 13b \end{bmatrix}$ is a linear transformation. Check:

- $T\left(\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} a' \\ b' \end{bmatrix}\right) = T\left(\begin{bmatrix} a+a' \\ b+b' \end{bmatrix}\right) = \begin{bmatrix} 2(a+a')+3(b+b') \\ 5(a+a')-7(b+b') \\ 11(a+a')+13(b+b') \end{bmatrix} = \begin{bmatrix} 2a+3b \\ 5a-7b \\ 11a+13b \end{bmatrix} + \begin{bmatrix} 2a'+3b' \\ 5a'-7b' \\ 11a'+13b' \end{bmatrix} = T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) + T\left(\begin{bmatrix} a' \\ b' \end{bmatrix}\right)$.
- $T\left(s\begin{bmatrix} a \\ b \end{bmatrix}\right) = T\left(\begin{bmatrix} sa \\ sb \end{bmatrix}\right) = \begin{bmatrix} 2sa+3sb \\ 5sa-7sb \\ 11sa+13sb \end{bmatrix} = \begin{bmatrix} s(2a+3b) \\ s(5a-7b) \\ s(11a+13b) \end{bmatrix} = s\begin{bmatrix} 2a+3b \\ 5a-7b \\ 11a+13b \end{bmatrix} = sT\left(\begin{bmatrix} a \\ b \end{bmatrix}\right)$.

NOTE: If $T : V \to U$ is a linear transformation, then $T(\vec{0}) = \vec{0}$.

Proof. Let T be linear. Then $T(\vec{0}) = T(0 \cdot \vec{0}) = 0 \cdot T(\vec{0}) = \vec{0}$. □

Example 7. The function $f : \mathbb{R}^2 \to \mathbb{R}^2$ satisfying $f\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} a+1 \\ b \end{bmatrix}$ is **NOT** linear.

We can tell this by observing that $T\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Example 8. The function $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined as $f\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} a^2 \\ b^2 \end{bmatrix}$ is **NOT** linear.

We can tell this by looking at the scalar rule:

$$T\left(s\begin{bmatrix} a \\ b \end{bmatrix}\right) = T\left(\begin{bmatrix} sa \\ sb \end{bmatrix}\right) = \begin{bmatrix} (sa)^2 \\ (sb)^2 \end{bmatrix} = s^2\begin{bmatrix} a^2 \\ b^2 \end{bmatrix} \neq s\begin{bmatrix} a^2 \\ b^2 \end{bmatrix} = sT\left(\begin{bmatrix} a \\ b \end{bmatrix}\right).$$

Example 9. Let A be the matrix $\begin{bmatrix} 2 & 3 \\ 5 & -7 \\ 11 & 13 \end{bmatrix}$. Then the function $T : \mathbb{R}^2 \to \mathbb{R}^3$ satisfying $T(\vec{v}) = A\vec{v}$ is a linear transformation. We can tell this by checking the two rules:

- $T(\vec{v} + \vec{u}) = A(\vec{v} + \vec{u}) = A\vec{v} + A\vec{u} = T(\vec{v}) + T(\vec{u})$.
- $T(s\vec{v}) = A(s\vec{v}) = sA\vec{v} = sT(\vec{v})$.

This is the same linear transformation as in Example 6!

Every linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ has a matrix associated to it. You can find what this matrix is by finding out what happens to the vectors of the standard basis.

Proposition 1. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, and let $\{\vec{e}_1, \ldots, \vec{e}_n\}$ be the standard basis for $\mathbb{R}^n$. Then the matrix for T is

$$A = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) & \cdots & T(\vec{e}_n) \end{bmatrix}.$$

That is, $T(\vec{v}) = A\vec{v}$ for every $\vec{v} \in \mathbb{R}^n$.

This is called the **standard basis method** of finding the matrix for T.

Example 10. Suppose $T : \mathbb{R}^3 \to \mathbb{R}^4$ is a linear transformation where we are given

$$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}6\\7\\8\\5\end{bmatrix}$$
$$T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}10\\-33\\15.5\\16\end{bmatrix}$$
$$T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}-4\\2\\1\\0.9\end{bmatrix}.$$

Then the matrix for T is

$$A = \begin{bmatrix}6 & 10 & -4\\7 & -33 & 2\\8 & 15.5 & 1\\5 & 16 & 0.9\end{bmatrix}.$$

When you have a basis $\{\vec{v}_1, \ldots, \vec{v}_\ell\}$ for a subspace $V \subseteq \mathbb{R}^n$, and you know the outputs $T(\vec{v}_1), \ldots, T(\vec{v}_\ell)$, then you can calculate $T(\vec{v})$ for any $\vec{v} \in V$ by writing $\vec{v}$ as a linear combination of the basis vectors

$$\vec{v} = s_1\vec{v}_1 + \cdots + s_\ell\vec{v}_\ell$$

and then evaluate $T(\vec{v})$ as

$$T(\vec{v}) = T(s_1\vec{v}_1 + \cdots s_\ell\vec{v}_\ell) = s_1T(\vec{v}_1) + \cdots s_\ell T(\vec{v}_\ell).$$

Example 11. Suppose $T : \mathbb{R}^3 \to \mathbb{R}^4$ is a linear transformation where we are given

$$T\left(\begin{bmatrix}2\\1\\1\end{bmatrix}\right) = \begin{bmatrix}4\\1\\5\\6\end{bmatrix}$$
$$T\left(\begin{bmatrix}3\\1\\0\end{bmatrix}\right) = \begin{bmatrix}0\\2\\7\\8\end{bmatrix}.$$

Then we have enough information to calculate $T\left(\begin{bmatrix}2\\3\\7\end{bmatrix}\right)$ but not $T\left(\begin{bmatrix}2\\3\\8\end{bmatrix}\right)$.

For the former, $\begin{bmatrix}2\\3\\7\end{bmatrix} = 7\begin{bmatrix}2\\1\\1\end{bmatrix} - 4\begin{bmatrix}3\\1\\0\end{bmatrix}$, so

$$T\left(\begin{bmatrix}2\\3\\7\end{bmatrix}\right) = T\left(7\begin{bmatrix}2\\1\\1\end{bmatrix} - 4\begin{bmatrix}3\\1\\0\end{bmatrix}\right) = 7T\left(\begin{bmatrix}2\\1\\1\end{bmatrix}\right) - 4T\left(\begin{bmatrix}3\\1\\0\end{bmatrix}\right) = 7\begin{bmatrix}4\\1\\5\\6\end{bmatrix} - 4\begin{bmatrix}0\\2\\7\\8\end{bmatrix} = \begin{bmatrix}28\\-1\\7\\10\end{bmatrix}.$$

On the contrary, $\begin{bmatrix}2\\3\\8\end{bmatrix} \notin \operatorname{span}\left\{\begin{bmatrix}2\\1\\1\end{bmatrix}, \begin{bmatrix}3\\1\\0\end{bmatrix}\right\}$, so we do not have enough information to find $T\left(\begin{bmatrix}2\\3\\8\end{bmatrix}\right)$.

So if you have a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ with a basis $\{\vec{v}_1, \ldots, \vec{v}_n\}$ for $\mathbb{R}^n$ for which you know the outputs $T(\vec{v}_1), \ldots, T(\vec{v}_n)$, then you can find a matrix for T by finding out what the outputs of the standard basis vectors are. To do this, write each standard basis vector $\vec{e}_i$ as a linear combination of the basis vectors you are given:

$$\vec{e}_i = s_{i,1}\vec{v}_i + \cdots + s_{i,n}\vec{v}_n.$$

Then you can calculate $T(\vec{e}_i)$ as

$$T(\vec{e}_i) = s_{i,1}T(\vec{v}_1) + \cdots + s_{i,n}T(\vec{v}_n)$$

and then use Proposition 1 to construct the matrix for T by using $T(\vec{e}_i)$ as the columns:

$$\begin{bmatrix}T(\vec{e}_1) & \cdots & T(\vec{e}_n)\end{bmatrix}.$$

Another way you can construct the matrix is by using Proposition 2:

Proposition 2. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Let $\{\vec{v}_1, \ldots, \vec{v}_n\}$ be any basis for $\mathbb{R}^n$. Then you can calculate the matrix for T by first constructing the matrix

$$A = \begin{bmatrix}T(\vec{v}_1) & \cdots & T(\vec{v}_n)\end{bmatrix}$$

whose columns are the outputs $T(\vec{v}_i)$ and the matrix

$$B = \begin{bmatrix}\vec{v}_1 & \cdots & \vec{v}_n\end{bmatrix}$$

whose columns are the basis vectors $\vec{v}_i$. Then the matrix for T is

$$AB^{-1}.$$

This is called the **change-of-basis** method of finding the matrix for T.

Example 12. Suppose $T : \mathbb{R}^3 \to \mathbb{R}^4$ is a linear transformation where we are given

$$T\left(\begin{bmatrix}2\\1\\1\end{bmatrix}\right) = \begin{bmatrix}4\\1\\5\\6\end{bmatrix}$$

$$T\left(\begin{bmatrix}3\\1\\0\end{bmatrix}\right) = \begin{bmatrix}0\\2\\7\\8\end{bmatrix}$$

$$T\left(\begin{bmatrix}0\\1\\2\end{bmatrix}\right) = \begin{bmatrix}1\\3\\4\\6\end{bmatrix}.$$

Note that $\left\{\begin{bmatrix}2\\1\\1\end{bmatrix}, \begin{bmatrix}3\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\2\end{bmatrix}\right\}$ is indeed a basis for $\mathbb{R}^3$. Then we can construct the matrix for T by first constructing

$$A = \begin{bmatrix}4 & 0 & 1\\1 & 2 & 3\\5 & 7 & 4\\6 & 8 & 6\end{bmatrix}$$

and

$$B = \begin{bmatrix}2 & 3 & 0\\1 & 1 & 1\\1 & 0 & 2\end{bmatrix}.$$

Then the matrix for T is

$$AB^{-1} = \begin{bmatrix}4 & 0 & 1\\1 & 2 & 3\\5 & 7 & 4\\6 & 8 & 6\end{bmatrix}\begin{bmatrix}2 & 3 & 0\\1 & 1 & 1\\1 & 0 & 2\end{bmatrix}^{-1} = \begin{bmatrix}4 & 0 & 1\\1 & 2 & 3\\5 & 7 & 4\\6 & 8 & 6\end{bmatrix}\begin{bmatrix}2 & -6 & 3\\-1 & 4 & -2\\-1 & 3 & -1\end{bmatrix} = \begin{bmatrix}7 & -21 & 11\\-3 & 11 & -4\\-1 & 10 & -3\\-2 & 14 & -4\end{bmatrix}.$$

This allows us to calculate

$$T\left(\begin{bmatrix}2\\3\\7\end{bmatrix}\right) = \begin{bmatrix}7 & -21 & 11\\-3 & 11 & -4\\-1 & 10 & -3\\-2 & 14 & -4\end{bmatrix}\begin{bmatrix}2\\3\\7\end{bmatrix} = \begin{bmatrix}28\\-1\\7\\10\end{bmatrix}$$ (which matches what we got in Example 11!)

and also

$$T\left(\begin{bmatrix}2\\3\\8\end{bmatrix}\right) = \begin{bmatrix}7 & -21 & 11\\-3 & 11 & -4\\-1 & 10 & -3\\-2 & 14 & -4\end{bmatrix}\begin{bmatrix}2\\3\\8\end{bmatrix} = \begin{bmatrix}39\\-5\\4\\6\end{bmatrix}.$$

Now we will move on to surjectivity and injectivity.

Proposition 3. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and let A be the $m \times n$ matrix associated to T. Then

$$\text{img}(T) = \text{img}(A)$$

where the img(T) is the image in the *function* sense (i.e. the image of a function is the set of all outputs) and the img(A) is the image in the *matrix* sense (i.e. the image of A is the set of all vectors $\vec{b} \in \mathbb{R}^m$ such that there is some vector $\vec{x} \in \mathbb{R}^n$ satisfying $A\vec{x} = \vec{b}$).

Recall that $\text{img}(A) = \text{col}(A)$, the column space of A (i.e., the vector space spanned by the columns of A). Additionally, recall that the dimension of the column space is equal to the rank of A. Thus we can investigate the dimension of the image of T (and by extension whether T is surjective or not) by calculating the rank of its matrix A.

Proposition 4. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and let A be the $m \times n$ matrix associated to T. Then T is surjective if and only if $\text{rank}(A) = m$ (the number of rows of A).

On the flip side, we can investigate the injectivity of A by looking at the nullity. Recall that for an $m \times n$ matrix A, the nullspace of A is $\text{null}(A) = \{\vec{v} \in \mathbb{R}^n : A\vec{v} = \vec{0}\}$, and $\text{nullity}(A) = \dim(\text{null}(A))$. For a linear function $T : \mathbb{R}^n \to \mathbb{R}^m$ with associated matrix A, the nullspace of A is called the kernel of T, denoted $\ker(T)$.

Proposition 5. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and let A be the $m \times n$ matrix associated to T. Then T is injective if and only if $\text{null}(A) = \{\vec{0}\}$ (i.e. $\text{nullity}(A) = 0$).

Proof. First, assume that T is injective. Then no two distinct inputs can have the same output. Since it follows quickly from the definition of linear functions that $T(\vec{0}) = \vec{0}$, we must have $\vec{0} \in \text{null}(A)$. Since T is injective, no other inputs may have an output of $\vec{0}$, so $\vec{0}$ must be the *only* thing in the nullspace. So $\text{null}(A) = \{\vec{0}\}$.

Now for the reverse direction, assume that $\text{null}(A) = \{\vec{0}\}$. We want to prove that this is sufficient for T to be injective. Let $\vec{u}, \vec{v} \in \mathbb{R}^n$ have the same outputs, so $T(\vec{u}) = T(\vec{v})$. We want to reach the conclusion that $\vec{u} = \vec{v}$. Since $T(\vec{u}) = T(\vec{v})$, then $T(\vec{u}) - T(\vec{v}) = \vec{0}$. Since T is linear, this means $T(\vec{u} - \vec{v}) = \vec{0}$, so $\vec{u} - \vec{v} \in \text{null}(A)$. Since we entered this argument with the assumption that $\text{null}(A) = \{\vec{0}\}$, we are forced to conclude that $\vec{u} - \vec{v} = \vec{0}$ and so $\vec{u} = \vec{v}$. Thus T is injective. □

At this point it is useful to restate the Rank-Nullity Theorem:

Theorem 1: Rank-Nullity. Let A be an $m \times n$ matrix. Then

$$\text{rank}(A) + \text{nullity}(A) = n \text{ (the number of columns).}$$

This means that whenever you know the rank of a matrix A, you are a very short calculation away from also knowing the nullity! (And vice-versa!) In other words, if you know whether a function is surjective, you are a very short calculation away from also knowing whether it is injective! (And vice-versa!)

Proposition 6. Let A be any $m \times n$ matrix and let B be an invertible $n \times n$ matrix. Then $\text{rank}(AB^{-1}) = \text{rank}(A)$ and $\text{nullity}(AB^{-1}) = \text{nullity}(A)$.

Example 13. Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be a linear transformation satisfying

$$T\left(\begin{bmatrix}2\\1\\1\end{bmatrix}\right) = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}$$

$$T\left(\begin{bmatrix}3\\1\\0\end{bmatrix}\right) = \begin{bmatrix}0\\1\\0\\0\end{bmatrix}$$

$$T\left(\begin{bmatrix}0\\1\\2\end{bmatrix}\right) = \begin{bmatrix}5\\9\\0\\0\end{bmatrix}$$

Then we can construct the matrix for T by setting up

$$A = \begin{bmatrix}1 & 0 & 5\\0 & 1 & 9\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$$

and

$$B = \begin{bmatrix}2 & 3 & 0\\1 & 1 & 1\\1 & 0 & 2\end{bmatrix}$$

and calculating

$$AB^{-1} = \begin{bmatrix}1 & 0 & 5\\0 & 1 & 9\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}2 & 3 & 0\\1 & 1 & 1\\1 & 0 & 2\end{bmatrix}^{-1} = \begin{bmatrix}1 & 0 & 5\\0 & 1 & 9\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}2 & -6 & 3\\-1 & 4 & -2\\-1 & 3 & -1\end{bmatrix} = \begin{bmatrix}-3 & 9 & -2\\-10 & 31 & -11\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}.$$

Then we can perform row operations on $\begin{bmatrix}-3 & 9 & -2\\-10 & 31 & -11\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$ to get it into row echelon form $\begin{bmatrix}1 & -3 & \frac{2}{3}\\0 & 1 & -\frac{13}{3}\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$ and calculate that the rank is 2. Therefore, the linear transformation T is NOT surjective because the rank would have to be 4 for T to be surjective.

Using Rank-Nullity, we can also calculate $\text{nullity}(A)$ using the equation

$$\text{rank}(A) + \text{nullity}(A) = 3$$

and solving for nullity$(A) = 1$. Therefore T is not injective either, because the nullity of A is not 0.

Note that because of Proposition 6, we can actually skip calculating B^{-1} and just use $A = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 9 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ instead, since $\text{rank}(A) = \text{rank}(AB^{-1})$. This is useful because A is already in reduced row echelon form, so we can automatically see that its rank is 2, and then use Rank-Nullity to find that its nullity is 1.

Proposition 7. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then:

- T CANNOT be surjective if $n < m$.
- T CANNOT be injective if $n > m$.
- If T is bijective (and therefore invertible), then $n = m$.

The converses of the above three statements are not true in general. That is,

- Just because $n \geq m$ does not necessarily mean T is surjective.
- Just because $n \leq m$ does not necessarily mean T is injective.
- Just because $n = m$ doe snot necessarily mean T is invertible.

A good example to keep in mind here is the so-called "zero transformation:" that is, a function $T : \mathbb{R}^n \to \mathbb{R}^m$ such that $T(\vec{v}) = \vec{0}$ for all $\vec{v} \in \mathbb{R}^n$. This is a linear transformation because

- $T(\vec{u} + \vec{v}) = \vec{0}$ and $T(\vec{u}) + T(\vec{v}) = \vec{0} + \vec{0} = \vec{0}$, so $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$,
- and $T(s\vec{v}) = \vec{0}$ and $sT(\vec{v}) = s\vec{0} = \vec{0}$, so $T(s\vec{v}) = sT(\vec{v})$.

But T cannot be injective or surjective for any positive numbers n and m, and so will also not be invertible.

Proposition 8. A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is invertible if and only if its associated linear transformation A has a non-zero determinant.

Definition 8. Let $T : V \to U$ be a bijective linear transformation. Then T is called an **isomorphism**. If two vector spaces V and U have an isomorphism between them, then V and U are said to be **isomorphic**.

Part 3: Special Linear Transformations

Two kinds of special linear transformations we will observe are *rotations* and *reflections*.

Definition 9. In $\mathbb{R}^2$, a rotation of the plane by an angle of θ in the counter-clockwise direction is a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$. It is given by the matrix

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$

Note that every rotation is invertible because the determinant of the above 2×2 matrix is $\cos^2(\theta) + \sin^2(\theta) = 1$. This makes intuitive sense too: you can always undo a rotation by rotating by the same angle in the other direction!

Definition 10. In $\mathbb{R}^2$, reflecting over the line $y = mx$ through the origin is a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$\frac{1}{m^2+1}\begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix}.$$

Note that the determinant of a reflection is

$$\frac{(1-m^2)(m^2-1)-4m^2}{(m^2+1)^2} = \frac{-1+2m^2-m^4-4m^2}{(m^2+1)^2} = -\frac{m^4+2m^2+1}{(m^2+1)^2} = -\frac{(m^2+1)^2}{(m^2+1)^2} = -1$$

so a reflection is also always invertible.

Also note that our definition of reflection does not include the possibility of flipping over the line $x = 0$ (the y-axis), which has a slope of infinity. To account for this, we can calculate

$$\lim_{m\to\infty}\begin{bmatrix} \frac{1-m^2}{m^2+1} & \frac{2m}{m^2+1} \\ \frac{2m}{m^2+1} & \frac{m^2-1}{m^2+1} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix},$$

or we can simply calculate the outputs for the standard basis vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$: the output for $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ when reflected over the y-axis is $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$ and the output for $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ when reflected over the y-axis is $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ (because this vector is on the y-axis, so it does not move). Putting these outputs together, we get the matrix $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ as before.

If $T : U \to V$ and $S : V \to W$ are linear transformations associated to matrices A and B respectively, you can find a matrix for the composition of functions $S \circ T$ as the product BA. Similarly, if $T : U \to V$ is an invertible linear transformation associated to the matrix A, then the matrix for the inverse function $T^{-1} : V \to U$ is given by A^{-1}.

Proposition 9. An orthogonal projection onto a vector $\vec{u} \in \mathbb{R}^2$ is a linear transformation.

Example 14. Let us consider the projection function onto the vector $\vec{u} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$. We can use two different methods to calculate the matrix for this function:

Method 1: The Standard Basis Method. (See Proposition 1) In order to compute the matrix for the linear transformation $\text{proj}_{\vec{u}} : \mathbb{R}^2 \to \mathbb{R}^2$, we can use the projection formula

$$\text{proj}_{\vec{u}}(\vec{v}) = \frac{\vec{v}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u}$$

to find the outputs of the standard basis vectors $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$. We have

$$\text{proj}_{\vec{u}}\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \frac{3}{13}\begin{bmatrix}3\\2\end{bmatrix} = \begin{bmatrix}9/13\\6/13\end{bmatrix},$$

and

$$\text{proj}_{\vec{u}}\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \frac{2}{13}\begin{bmatrix}3\\2\end{bmatrix} = \begin{bmatrix}6/13\\4/13\end{bmatrix}.$$

Putting these outputs together as columns of a matrix, we get

$$A = \begin{bmatrix}9/13 & 6/13\\6/13 & 4/13\end{bmatrix}.$$

Method 2: The Change-of-Basis Method. (See Proposition 2) We can also forgo using the projection formula and instead rely on the geometric principles of projection to calculate the matrix for $\text{proj}_{\vec{u}}$. For example, since we are projecting onto the vector $\vec{u} = \begin{bmatrix}3\\2\end{bmatrix}$, it must be the case that

$$\text{proj}_{\vec{u}}\left(\begin{bmatrix}3\\2\end{bmatrix}\right) = \begin{bmatrix}3\\2\end{bmatrix},$$

in other words, $\begin{bmatrix}'3\\2\end{bmatrix}$ is "fixed" by the function. Additionally, any vector perpendicular to the line of projection must have an output of $\vec{0}$. For example,

$$\text{proj}_{\vec{u}}\left(\begin{bmatrix}-2\\3\end{bmatrix}\right) = \begin{bmatrix}0\\0\end{bmatrix},$$

since $\begin{bmatrix}-2\\3\end{bmatrix}$ is orthogonal to $\begin{bmatrix}3\\2\end{bmatrix}$. Then we can construct a matrix out of the outputs $\begin{bmatrix}3\\2\end{bmatrix}$ and $\begin{bmatrix}0\\0\end{bmatrix}$,

$$M = \begin{bmatrix}3 & 0\\2 & 0\end{bmatrix}$$

and a change-of-basis matrix

$$B = \begin{bmatrix}3 & -2\\2 & 3\end{bmatrix}$$

made from the basis vectors $\left\{\begin{bmatrix}3\\2\end{bmatrix}, \begin{bmatrix}-2\\3\end{bmatrix}\right\}$. Then

$$A = MB^{-1}$$

is the matrix for proj$\vec{u}$.

We can calculate to make sure that our answer using Method 2 matches our answer using Method 1.

$$B^{-1} = \frac{1}{13}\begin{bmatrix} 3 & 2 \\ -2 & 3 \end{bmatrix},$$

so

$$MB^{-1} = \frac{1}{13}\begin{bmatrix} 3 & 0 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ -2 & 3 \end{bmatrix} = \frac{1}{13}\begin{bmatrix} 9 & 6 \\ 6 & 4 \end{bmatrix} = \begin{bmatrix} 9/13 & 6/13 \\ 6/13 & 4/13 \end{bmatrix},$$

just like in Method 1.

Notice here that the determinant of A is 0, meaning that A is not invertible. This lines up geometrically with projection, since projection is not an invertible function! For example, projection is not injective since every vector perpendicular to the line of projection shares the same output of $\vec{0}$. Neither is projection surjective, since the image of $\text{proj}_{\vec{u}}$ is the line $\langle \vec{u} \rangle$, but the codomain is $\mathbb{R}^2$, which are not equal.

Part 4: Spectral Theory

Definition 11. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation with associated $n \times n$ matrix A. An **eigenvector** of A is a nonzero vector $\vec{v}$ such that $A\vec{v} = s\vec{v}$ for some scalar $s \in \mathbb{R}$. In this case, s is the **eigenvalue** for $\vec{v}$.

Example 15. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by the matrix

$$A = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}.$$

Then

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

so $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eigenvector of A and 2 is its eigenvalue. Additionally,

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -3 \end{bmatrix} = -3\begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

so $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is an eigenvector and -3 is its eigenvalue.

Note that any nonzero multiple of an eigenvector is another eigenvector with the same eigenvalue. For example,

$$T\left(\begin{bmatrix} 18 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 36 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 18 \\ 0 \end{bmatrix},$$

so $\begin{bmatrix} 18 \\ 0 \end{bmatrix}$ is also an eigenvector with an eigenvalue of 2. But not necessarily will every vector be an eigenvector. For example $\begin{bmatrix} 4 \\ 5 \end{bmatrix}$ is not an eigenvector because

$$T\left(\begin{bmatrix} 4 \\ 5 \end{bmatrix}\right) = \begin{bmatrix} 8 \\ -15 \end{bmatrix},$$

and $\begin{bmatrix} 8 \\ -15 \end{bmatrix}$ is not a scalar multiple of $\begin{bmatrix} 4 \\ 5 \end{bmatrix}$.

Definition 12. Let A be a square matrix. The set of all eigenvalues of A is called the **spectrum** of A.

To find the eigenvalues of an $n \times n$ matrix A, first note that s is an eigenvalue if there is some nonzero vector $\vec{v}$ such that

$$A\vec{v} = s\vec{v}.$$

Inserting the identity matrix I_n into the equation gives us

$$A\vec{v} = sI_n\vec{v}$$

(we can do this because multiplying by the identity matrix I_n doesn't actually change the result, but we will need it when we modify the equation later). Subtracting $sI_n\vec{v}$ from both sides yields

$$A\vec{v} - sI_n\vec{v} = \vec{0},$$

and now we can factor $\vec{v}$ from both terms and get

$$(A - sI_n)\vec{v} = \vec{0}$$

(this is where the inclusion of the I_n is important: $A - sI_n$ is the difference of two matrices, which is valid; but just $A - s$ is the difference of a matrix and a number, which is not valid).

From this last equation, we have $\vec{v} \in \text{null}(A - sI_n)$, but remember eigenvectors are nonzero! So that means the matrix $A - sI_n$ has something nonzero in its nullspace, which means it cannot be injective and so it cannot be invertible! In other words,

$$\det(A - sI_n) = 0.$$

So to find eigenvalues of A, we need to find what values of make the determinant of $A - sI_n$ zero.

Definition 13. Let A be an $n \times n$ matrix and let s be a variable, and I_n is the $n \times n$ identity matrix. Then

$$\det(A - sI_n)$$

is a degree-n polynomial. This is called the **characteristic polynomial** of A, and it is denoted $c_A(s)$.

Thus the eigenvalues of A are the *roots* of the characteristic polynomial of A.

Example 15 continued. Let's return to the matrix $A = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$ and calculate $c_A(s)$. We have

$$A - sI_2 = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix} - \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} = \begin{bmatrix} 2-s & 0 \\ 0 & -3-s \end{bmatrix},$$

and so the determinant (and thus $c_A(s)$) is $(2-s)(-3-s)$. The roots of this polynomial are $s = 2$ and $s = -3$, which match the eigenvalues we got earlier.

When finding eigenvalues, the best kind of matrix to encounter is a *diagonal matrix*:

Definition 12. A **diagonal matrix** is a square matrix D where $D_{ij} = 0$ for all entries where $i \neq j$ (i.e. for entries where the row number is different from the column number). In other words, D is diagonal if every entry that is not on the "**main diagonal**" of D is 0.

For example, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -14 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 6 & 0 & 0 & 0 \\ 0 & 7 & 0 & 0 \\ 0 & 0 & -11 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ are all diagonal matrices.

Proposition 10. Let D be a diagonal matrix. Then the eigenvalues for D are the entries on the main diagonal of D.

For example, the eigenvalues of $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -14 \end{bmatrix}$ are 1, 7, and -14, the eigenvalues of $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, are just 0, and the eigenvalues of $\begin{bmatrix} 6 & 0 & 0 & 0 \\ 0 & 7 & 0 & 0 \\ 0 & 0 & -11 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ are 6, 7, -11, and 0.

Example 16. Let $A = \begin{bmatrix} 7 & 5 \\ -10 & -8 \end{bmatrix}$. Since A is not diagonal, we will have to use the characteristic polynomial to find the eigenvalues of A. We get

$$A - sI_2 = \begin{bmatrix} 7-s & 5 \\ -10 & -8-s \end{bmatrix},$$

which has a determinant of

$$(7-s)(-8-s) + 50 = -56 - 7s + 8s + s^2 + 50 = s^2 + s - 6 = (s-2)(s+3).$$

So the roots of the characteristic polynomial are $s = 2$ and $s = -3$.

Now we can go another step and find the eigenvectors: to do this, we need to plug in our eigenvalues for s in $\begin{bmatrix} 7-s & 5 \\ -10 & -8-s \end{bmatrix}$ and find the nullspaces of the resulting matrices.

We will begin with the eigenvalue of 2. We want to find a solution to the augmented matrix

$$\left[\begin{array}{cc|c} 5 & 5 & 0 \\ -10 & -10 & 0 \end{array}\right].$$

Getting this into reduced row echelon form, we have

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right],$$

and so (using x and y as variables for the first and second columns, respectively) we have x as the basic variable and y as the free variable. Therefore we can parametrize y with the parameter t and then the first row tells us $x + t = 0$, so $x = -t$. Thus the parametrization for the solution set is

$$\begin{aligned} x &= -t \\ y &= t, \end{aligned}$$

or written as vectors,

$$\begin{bmatrix} x \\ y \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Therefore any scalar multiple of $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is an eigenvector for the eigenvalue of 2. Since we are just interested in finding one, we can just pick $t = 1$ and use $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ as our eigenvector. We can confirm this by multiplying

$$\begin{bmatrix} 7 & 5 \\ -10 & -8 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -7+5 \\ 10-8 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Now for the eigenvalue of -3, we get

$$\left[\begin{array}{cc|c} 10 & 5 & 0 \\ -10 & -5 & 0 \end{array}\right],$$

and using row operations we get the reduced row echelon form

$$\left[\begin{array}{cc|c} 1 & 1/2 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

So we get again x is basic and y is free: thus we parametrize y as $y = t$ and use the first row to get the equation $x + \frac{1}{2}t = 0$, so the parametrization is

$$\begin{aligned} x &= -\frac{1}{2}t \\ y &= t, \end{aligned}$$

or written as vectors,

$$\begin{bmatrix} x \\ y \end{bmatrix} = t \begin{bmatrix} -1/2 \\ 1 \end{bmatrix}.$$

So any scalar multiple of $\begin{bmatrix} -1/2 \\ 1 \end{bmatrix}$ is an eigenvector for the eigenvalue of -3. Since we're just interested in finding one, we can set $t = 2$ to get rid of the fractions and get $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ as our eigenvector for the eigenvalue -3. We can confirm this by multiplying

$$\begin{bmatrix} 7 & 5 \\ -10 & -8 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -7+10 \\ 10-16 \end{bmatrix} = \begin{bmatrix} 3 \\ -6 \end{bmatrix} = -3 \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Note in this example, $A = \begin{bmatrix} 7 & 5 \\ -10 & -8 \end{bmatrix}$ has the same eigenvalues as the diagonal matrix $D = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$, but only the eigenvectors are different. In fact, using the eigenvectors $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ we found earlier, we can be more specific about the relationship between A and D! Putting the eigenvectors together in the matrix $P = \begin{bmatrix} -1 & -1 \\ 1 & 2 \end{bmatrix}$, we find $A = PDP^{-1}$. To understand where this relationship comes from, let's break down the product

$$\underbrace{\begin{bmatrix} -1 & -1 \\ 1 & 2 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} -1 & -1 \\ 1 & 2 \end{bmatrix}^{-1}}_{P^{-1}}$$

in detail, by analyzing

$$\begin{bmatrix} -1 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$
$$\text{and } \begin{bmatrix} -1 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$

As a function, P sends an input of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to an output of $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. That means that P^{-1} reverses that direction: P^{-1} receives an input of $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and outputs $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Then next D receives $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ as an input and outputs $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ as a result. Finally, P receives an input of $\begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and outputs $2\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$. So all together, PDP^{-1} takes an input of $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and outputs $\begin{bmatrix} -2 \\ 2 \end{bmatrix}$, just like A does!

A similar analysis reveals that PDP^{-1} takes an input of $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ to an output of $\begin{bmatrix} 3 \\ -6 \end{bmatrix}$ (because P^{-1} carries $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ to $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, which D carries to $\begin{bmatrix} 0 \\ -3 \end{bmatrix}$, which P carries to $\begin{bmatrix} 3 \\ -6 \end{bmatrix}$). This again matches A. This is sufficient to confirm the equation $A = PDP^{-1}$.

The specific name for this relationship between A and D is called *similarity*:

Definition 13. Let A and B be square matrices. Then A is **similar** to B if there is an invertible matrix P such that $A = PBP^{-1}$. The sentence "A is similar to B" can be written in shorthand as $A \sim B$.

Proposition 11. Let A, B and C be square matrices. Then the following are always true:

- $A \sim A$ (Reflexive).

- If $A \sim B$ then $B \sim A$ (Symmetric).
- If $A \sim B$ and $B \sim C$, then $A \sim C$ (Transitive).

Furthermore, suppose $A \sim B$. Then:

- $\det(A) = \det(B)$.
- $c_A(s) = c_B(s)$.
- A and B have the same eigenvalues.
- $\text{rank}(A) = \text{rank}(B)$.
- $\text{nullity}(A) = \text{nullity}(B)$.

Knowing that a matrix A is similar to a diagonal matrix is very useful, because a lot of the above information is easy to compute for a diagonal matrix, and that information can then be carried back to A.

Definition 14. A square matrix A is called **diagonalizable** if it is similar to some diagonal matrix D. In this case, D is then called a **diagonalization** of A.

So returning to Example 15, we have

$$\begin{bmatrix} 7 & 5 \\ -10 & -8 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix},$$

so $\begin{bmatrix} 7 & 5 \\ -10 & -8 \end{bmatrix}$ is *diagonalizable* and $\begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$ is a *diagonalization* of $\begin{bmatrix} 7 & 5 \\ -10 & -8 \end{bmatrix}$.

Diagonal matrices are nice because their powers are easy to calculate: if D is diagonal, then the (i,j) entry of D^n is the n^{th} power of the (i,j) entry of D. To put it another way, $[D^n]_{ij} = [D_{ij}]^n$.

For example, $\begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}^{10} = \begin{bmatrix} 2^{10} & 0^{10} \\ 0^{10} & (-3)^{10} \end{bmatrix} = \begin{bmatrix} 1024 & 0 \\ 0 & 59049 \end{bmatrix}$.

Proposition 12. Let $A = PBP^{-1}$. Then $A^n = PB^nP^{-1}$, for any n.

Proof. This will be a good introduction to what is known as a *proof by induction.* A proof by induction breaks into two basic steps: known as the "base case" and the "inductive step."

The base case is about proving what you want to prove for a specific value n. You want this value of n to be low, so we will prove that the claim is true for $n = 2$, as follows:

- Base Case: We will prove the claim for $n = 2$. That is, we will prove that $A^2 = PB^2P^{-1}$. We get

$$A^2 = (PBP^{-1})^2 = PBP^{-1}PBP^{-1} = PBBP^{-1} = PB^2P^{-1}.$$

  Thus the claim is true for when $n = 2$, so we are done with the Base Case.

- Inductive Step: The point of the inductive step is to prove that the claim is true for every value of n bigger than the one in the Base Case. For this, we will start by assuming the "inductive hypothesis:" the inductive hypothesis is that the claim is *already true* for one value of n, and we just want to prove it is true for $n+1$.

  Specifically, we will assume that $A^n = PB^nP^{-1}$ for a generic value n, and we want to use the assumption that $A^n = PB^nP^{-1}$ to prove that $A^{n+1} = PB^{n+1}P^{-1}$. To do this, we can write

$$A^{n+1} = (PBP^{-1})^{n+1} = (PBP^{-1})^n(PBP^{-1})$$
$$= \underbrace{(PB^nP^{-1})}_{\text{inductive hypothesis}} (PBP^{-1}) = PB^nBP^{-1} = PB^{n+1}P^{-1}.$$

  And so we know that $A^{n+1} = PB^{n+1}P^{-1}$ from the assumption that $A^n = PB^nP^{-1}$.

Together, the Base Case and the Inductive Step prove that the claim is true for any value of $n \geq 2$. This is because the Base Case proved that the claim is true for when $n = 2$, and the Inductive Step tells us that because the claim is true when $n = 2$, it must be true for $2+1$. And then since the claim is true for 3, the Inductive Step says it must be true for $3+1$ and so on. □

Example 17. Consider the matrix $A = \begin{bmatrix} 19 & 232 & -190 \\ 11 & 155 & -125 \\ 15 & 208 & -168 \end{bmatrix}$. We want to calculate A^{100}. The eigenvalues of A are 1, 2, and 3. In fact, we can write the diagonalization of A as

$$\underbrace{\begin{bmatrix} 19 & 232 & -190 \\ 11 & 155 & -125 \\ 15 & 208 & -168 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 13 & 10 & 4 \\ 8 & 5 & 3 \\ 11 & 7 & 4 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} -1 & -12 & 10 \\ 1 & 8 & -7 \\ 1 & 19 & -15 \end{bmatrix}}_{P^{-1}}.$$

Then

$$A^{100} = PD^{100}P^{-1}.$$

Then since D is diagonal, D^{100} is relatively easy to calculate:

$$D^{100} = \begin{bmatrix} 1^{100} & 0 & 0 \\ 0 & 2^{100} & 0 \\ 0 & 0 & 3^{100} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2^{100} & 0 \\ 0 & 0 & 3^{100} \end{bmatrix}.$$

Then

$$A^{100} = \begin{bmatrix} 13 & 10 & 4 \\ 8 & 5 & 3 \\ 11 & 7 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2^{100} & 0 \\ 0 & 0 & 3^{100} \end{bmatrix} \begin{bmatrix} -1 & -12 & 10 \\ 1 & 8 & -7 \\ 1 & 19 & -15 \end{bmatrix}$$
$$= \begin{bmatrix} 10*2^{100} + 4*3^{100} - 13 & 80*2^{100} + 76*3^{100} - 156 & -70*2^{100} - 60*3^{100} + 130 \\ 5*2^{100} + 3*3^{100} - 8 & 40*2^{100} + 57*3^{100} - 96 & -35*2^{100} - 45*3^{100} + 80 \\ 7*2^{100} + 4*3^{100} - 11 & 56*2^{100} + 76*3^{100} - 132 & -49*2^{100} - 60*3^{100} + 110 \end{bmatrix},$$

which is easier than multiplying A by itself one hundred times (both for people and for computer programs!).

Definition 15. Let A be an $n \times n$ matrix and let λ ("lambda") be an eigenvalue of A. Then the **eigenspace** of λ, denoted $E_\lambda(A)$, is the nullspace of $A - \lambda I_n$:

$$E_\lambda(A) = \text{null}(A - \lambda I_n).$$

Example 16 continued. For $A = \begin{bmatrix} 7 & 5 \\ -10 & -8 \end{bmatrix}$, we have the two eigenvalues 2 and -3. The eigenspace for the eigenvalue of 2 is the nullspace of $A - 2I_2 = \begin{bmatrix} 5 & 5 \\ -10 & -10 \end{bmatrix}$. We already solved this system of equations when we found the eigenvector in Example 15. The nullspace of $A - 2I_2$ is $\text{span}\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$. Therefore

$$E_2(A) = \text{span}\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}.$$

Similarly,

$$E_{-3}(A) = \text{null}(A + 3I_2) = \text{null}\left(\begin{bmatrix} 10 & 5 \\ -10 & -5 \end{bmatrix} \right) = \text{span}\left\{ \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right\}.$$

So in summary, the eigenspace of λ is the set of *all* eigenvectors for λ for the matrix A.

Definition 16. Let $f(x)$ be a polynomial, and let r be a root of f (i.e. $f(r) = 0$). The **multiplicity** of r as a root of f is the largest number m such that $f(x)$ is divisible by $(x - r)^m$. In other words, the largest m such that

$$\frac{f(x)}{(x - r)^m}$$

is a polynomial.

Example 18. Let's consider the polynomial $f(x) = (x - 2)^3(x - 4)^6(x + 1)$. In this case, $f(x)$ is divisible by $(x - 2)^3$ but not by $(x - 2)^4$, so the multiplicity of 2 as a root of f is 3. Similarly, $f(x)$ is divisible by $(x - 4)^6$ but not by $(x - 4)^7$, so the multiplicity of 4 as a root of f is 6. And since $f(x)$ is divisible by $(x + 1)$ but not by $(x + 1)^2$, the multiplicity of -1 as a root of f is 1.

When your polynomial is completely factored (as in the above example), it is straightforward to find the multiplicity of each root, as the multiplicities are equal to the numbers in each exponent. In other cases, there is an **alternative method** you can use to determine the multiplicity of a root.

Proposition 13. Let $f(x)$ be a polynomial, and let r be a root of f. Then the multiplicity of r as a root of f is the *smallest* number m such that

$$f^{(m)}(r) \neq 0,$$

where $f^{(m)}$ is the m^{th} order derivative, as from calculus.

Example 19. Let's see this work with a known example: let $f(x) = (x-2)^3$. We can tell just by the exponent that the multiplicity of the root $x = 2$ is 3. But if we were to apply Proposition 12, we get

$$\begin{aligned} f(2) &= (2-2)^3 = 0 \\ f'(2) &= 3(2-2)^2 = 0 \\ f''(2) &= 6(2-2) = 0 \\ f'''(2) &= 6 \neq 0, \end{aligned}$$

so we need to take the derivative of $f(x)$ three times for 2 to stop being a root. So according to Proposition 12, the multiplicity of 2 as a root of $f(x)$ is 3, which is good because that matches what we already knew.

Now we will return to characteristic polynomials:

Proposition 14. Let A be an $n \times n$ matrix and let $c_A(s)$ be the characteristic polynomial of A. Let λ be a root of the characteristic polynomial of multiplicity m_λ. Then

$$1 \leq \dim E_\lambda(A) \leq m_\lambda,$$

where again $E_\lambda(A)$ is the eigenspace of λ.

Furthermore, A is only diagonalizable if

$$\dim E_\lambda(A) = m_\lambda$$

for <u>all</u> eigenvalues λ.

Example 20. Let $A = \begin{bmatrix} -1 & 6 & -10 \\ 0 & 5 & -10 \\ 0 & 3 & -6 \end{bmatrix}$. To find out if A is diagonalizable, first we need to find its eigenvalues. Take the characteristic polynomial

$$\begin{aligned} c_A(s) &= \det\left(\begin{bmatrix} -1-s & 6 & -10 \\ 0 & 5-s & -10 \\ 0 & 3 & -6-s \end{bmatrix}\right) \\ &= (-1-s)((5-s)(-6-s)+30) - 6(0) - 10(0) = -s^3 - 2s^2 - s. \end{aligned}$$

This polynomial can be factored as

$$c_A(s) = -s(s^2 + 2s + 1) = -s(s+1)^2,$$

which has one root at $s = 0$ of multiplicity 1 and another root at $s = -1$ of multiplicity 2. So now we know that

$$\dim E_{-1}(A) \leq 2,$$

and A will only be diagonalizable if $\dim E_{-1}(A) = 2$. To figure out $\dim E_{-1}(A)$, we need to compute nullity$(A + I_3)$. We can set up the augmented matrix

$$\left[\begin{array}{ccc|c} 0 & 6 & -10 & 0 \\ 0 & 6 & -10 & 0 \\ 0 & 3 & -5 & 0 \end{array}\right].$$

Putting this in reduced row echelon form, we get

$$\left[\begin{array}{ccc|c} 0 & 1 & -5/3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Labeling the columns x, y, and z, we have one basic variable (y) and two free variables (x and z). So we can parametrize $x = t$ and $z = u$ and use the first row of the matrix to give us the equation $y - \frac{5}{3}z = 0$, so $y = \frac{5}{3}u$. So our full parametrization is

$$\begin{aligned} x &= t \\ y &= \frac{5}{3}u \\ z &= u, \end{aligned}$$

or written as a vector equation:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + u \begin{bmatrix} 0 \\ 5/3 \\ 1 \end{bmatrix}.$$

Therefore the nullspace of $A+I_3$ is two-dimensional, spanned by the vectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 5/3 \\ 1 \end{bmatrix}$. Since the nullspace of $A + I_3$ is $E_{-1}(A)$, we can confirm that the eigenspace $E_{-1}(A)$ is two-dimensional, and

$$E_{-1}(A) = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 5/3 \\ 1 \end{bmatrix} \right\}.$$

If we want to use a basis without any fractions, we can make an adjustment:

$$E_{-1}(A) = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ 3 \end{bmatrix} \right\}.$$

So we know that $\dim E_{-1}(A) = 2$. We also know from Proposition 13 that for the eigenvalue of 0, $1 \leq \dim E_0(A) \leq 1$, so $E_0(A) = 1$. Therefore we know that A is diagonalizable, since the dimension of each eigenspace is equal to the multiplicity of its corresponding eigenvalue.

Specifically, a diagonalization for A is $D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Then in order to find the matrix P such that $A = PDP^{-1}$, we will need a basis for each eigenspace. We already have a basis for the eigenspace $E_{-1}(A)$. So we just need to find a basis for $E_0(A)$. We need to find the nullspace of $A - 0I_3 = A$. We can set up the augmented matrix

$$\left[\begin{array}{ccc|c} -1 & 6 & -10 & 0 \\ 0 & 5 & -10 & 0 \\ 0 & 3 & -6 & 0 \end{array}\right]$$

and get it into reduced row echelon form

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

So we have two basic variables (x and y) and one free variable (z). Then we can parametrize $z = t$ and use the first and second rows to solve for x and y and get

$$\begin{aligned} x &= 2t \\ y &= 2t \\ z &= t, \end{aligned}$$

which yields the vector equation

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix},$$

so

$$E_0(A) = \text{null}(A) = \text{span}\left\{ \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \right\}.$$

So we can use the bases

$$E_{-1}(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ 3 \end{bmatrix} \right\} \text{ and } E_0(A) = \text{span}\left\{ \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \right\}$$

to set up the matrix

$$P = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 5 & 2 \\ 0 & 3 & 1 \end{bmatrix}.$$

Then

$$\underbrace{\begin{bmatrix} -1 & 6 & -10 \\ 0 & 5 & -10 \\ 0 & 3 & 6 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 0 & 2 \\ 0 & 5 & 2 \\ 0 & 3 & 1 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1 & 0 & 2 \\ 0 & 5 & 2 \\ 0 & 3 & 1 \end{bmatrix}^{-1}}_{P^{-1}}.$$

Note that the order of the columns matters here a little: The $\begin{bmatrix}1\\0\\0\end{bmatrix}$ and $\begin{bmatrix}0\\5\\3\end{bmatrix}$ columns need to match the positions of the $\begin{bmatrix}-1\\0\\0\end{bmatrix}$ and $\begin{bmatrix}0\\-1\\0\end{bmatrix}$ columns (but in either order), and the $\begin{bmatrix}2\\2\\1\end{bmatrix}$ column needs to match the position of the $\begin{bmatrix}0\\0\\0\end{bmatrix}$ column. So another valid diagonalization for A is

$$\begin{bmatrix}-1 & 6 & -10\\ 0 & 5 & -10\\ 0 & 3 & 6\end{bmatrix} = \begin{bmatrix}0 & 1 & 2\\ 5 & 0 & 2\\ 3 & 0 & 1\end{bmatrix}\begin{bmatrix}-1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}0 & 1 & 2\\ 5 & 0 & 2\\ 3 & 0 & 1\end{bmatrix}^{-1}$$

or even

$$\begin{bmatrix}-1 & 6 & -10\\ 0 & 5 & -10\\ 0 & 3 & 6\end{bmatrix} = \begin{bmatrix}0 & 2 & 1\\ 5 & 2 & 0\\ 3 & 1 & 0\end{bmatrix}\begin{bmatrix}-1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{bmatrix}\begin{bmatrix}0 & 2 & 1\\ 5 & 2 & 0\\ 3 & 1 & 0\end{bmatrix}^{-1}.$$

Example 21. Consider the matrix $A = \begin{bmatrix}1 & -7 & 12\\ 0 & -5 & 10\\ 0 & -3 & 6\end{bmatrix}$. To find out if A is diagonalizable, first we need to find its eigenvalues. Take the characteristic polynomial

$$c_A(s) = \det\left(\begin{bmatrix}1-s & -7 & 12\\ 0 & -5-s & 10\\ 0 & -3 & 6-s\end{bmatrix}\right) = (1-s)((-5-s)(6-s)+30) = -s^3+2s^2-s$$

which factors as

$$-s(s^2-2s+1) = -s(s-1)^2,$$

which has one root at $s=0$ of multiplicity 1 and another root at $s=1$ of multiplicity 2. So we know that

$$\dim E_1(A) \leq 2,$$

but A will only be diagonalizable if $\dim E_1(A) = 2$. To find $E_1(A)$, we need to find the nullspace of $A - I_3$. We can set up the augmented matrix

$$\left[\begin{array}{ccc|c}0 & -7 & 12 & 0\\ 0 & -6 & 10 & 0\\ 0 & -3 & 5 & 0\end{array}\right],$$

which has reduced row echelon form

$$\left[\begin{array}{ccc|c}0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\end{array}\right],$$

which has two basic variables (y and z) and one free variable (x). So we can parametrize $x = t$ and use the first and second rows to solve for y and z: we get $y = 0$ and $z = 0$. So the full parametrization of the nullspace is

$$\begin{aligned} x &= t \\ y &= 0 \\ z &= 0, \end{aligned}$$

or

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

So $E_1(A)$ is one-dimensional, because

$$E_1(A) = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

This means that A is not diagonalizable, since $\dim E_1(A) < 2$.

Part 5: Markov Matrices and Other Applications

Definition 17. A **Markov matrix** is a square matrix A such that

- A has no negative entries,
- The sum of the each column's entries is 1.

Proposition 15. Let A be an $n \times n$ Markov matrix. Then A has an eigenvalue of 1.

Proof. To show that A has an eigenvalue of 1, we need to show that $\det(A - I_n) = 0$. Since A is a Markov matrix, we know that the entries of each column add up to 1. Let

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{bmatrix}$$

where

$$\sum_{i=1}^{n} a_{i,j} = 1$$

for all j (i.e. each column's entries add to 1). Then

$$A - I_n = \begin{bmatrix} a_{1,1} - 1 & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} - 1 & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} - 1 \end{bmatrix}.$$

Now the sum of all the entries of column j of $A - I_n$ is

$$\left(\sum_{i=1}^{n} a_{i,j}\right) - 1 = 1 - 1 = 0,$$

so all the entries of the columns of $A - I_n$ add up to 0. So if we consider each column of $A - I_n$ as an individual vector,

$$\vec{v}_1 = \begin{bmatrix} a_{1,1} - 1 \\ a_{2,1} \\ \vdots \\ a_{n,1} \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} a_{1,2} \\ a_{2,2} - 1 \\ \vdots \\ a_{n,2} \end{bmatrix}, \cdots, \vec{v}_n = \begin{bmatrix} a_{1,n} \\ a_{2,n} \\ \vdots \\ a_{n,n} - 1 \end{bmatrix},$$

we see that all the entries of the vectors $\vec{v}_j$ satisfy the homogeneous linear equation

$$x_1 + x_2 + \cdots + x_n = 0$$

because all the entries in each column add up to 0. As we learned in the very very beginning of the semester, the solution sets of linear equations are $n-1$-dimensional hyperplanes in $\mathbb{R}^n$. And since this linear equation is homogenous, its corresponding hyperplane contains the origin, so it is a vector space! Specifically, the hyperplane is the nullspace of the matrix

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix},$$

which yields 1 basic variable and $n-1$ free variables.

So this means that the n vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are all in an $n-1$-dimensional vector space together. Therefore $\dim \operatorname{span}\{\vec{v}_1, \ldots, \vec{v}_n\} \leq n - 1 < n$, so the vectors $\vec{v}_1, \ldots, \vec{v}_n$ are linearly dependent. Since the $\vec{v}_j$ vectors are the columns of $A - I_n$, this means that $\operatorname{rank}(A - I_n) < n$, so $A - I_n$ is not surjective, so $A - I_n$ is not invertible, so $\det(A - I_n) = 0$. Thus 1 is an eigenvalue of A. □

Remark 1. Note that by the argument above, you can also say that if A is an $n \times n$ matrix such that there is a number λ such that

$$\sum_{i=1}^{n} a_{i,j} = \lambda$$

for all j (i.e. the sum of each column's entries is λ), then λ is an eigenvalue of A. Similarly, if

$$\sum_{j=1}^{n} a_{i,j} = \mu$$

for all i (i.e. the sum of each row's entries is μ), then μ is an eigenvector of A.

One application of Markov matrices is to model migration patterns in a closed system.

Example 22. Suppose we know the following information about three locations: L1, L2, and L3.

- 40% of the individuals in L1 in year t will remain in L1 in year $t+1$.
- 30% of the individuals in L1 in year t will move to L2 in year $t+1$.
- 30% of the individuals in L1 in year t will move to L3 in year $t+1$.
- 0% of the individuals in L2 in year t will move to L1 in year $t+1$.
- 90% of the individuals in L2 in year t will remain in L2 in year $t+1$.
- 10% of the individuals in L2 in year t will move to L3 in year $t+1$.
- 80% of the individuals in L3 in year t will move to L1 in year $t+1$.
- 10% of the individuals in L3 in year t will move to L2 in year $t+1$.
- 10% of the individuals in L3 in year t will remain in L3 in year $t+1$.

We can compile all this information into a Markov matrix

$$A = \begin{bmatrix} 0.4 & 0 & 0.8 \\ 0.3 & 0.9 & 0.1 \\ 0.3 & 0.1 & 0.1 \end{bmatrix}$$

where each the (i, j) entry of A reports what proportion of the population of location j will move to location i each year.

Definition 18. A **state vector** $\vec{x}_t$ reports the state of a system being modeled by a matrix A at time step t, where

$$\vec{x}_t = A\vec{x}_{t-1},$$

or alternatively

$$\vec{x}_t = A^t\vec{x}_0.$$

In our example, a state vector could look like $\vec{x}_0 = \begin{bmatrix} 1000 \\ 1600 \\ 4000 \end{bmatrix}$, which indicates that at time step $t = 0$, the population of L1 is 1000, the population of L2 is 1600, and the population at L3 is 4000.

Definition 19. A **steady state vector** $\vec{x}_s$ for a matrix A is a state vector such that

$$A\vec{x}_s = \vec{x}_s,$$

in other words $\vec{x}_s$ is an eigenvector for the eigenvalue of 1. Recall that Markov matrices always have an eigenvalue of 1 by Proposition 14, and so Markov matrices always have steady state vectors.

Example 22 continued. Let's bring back $A = \begin{bmatrix} 0.4 & 0 & 0.8 \\ 0.3 & 0.9 & 0.1 \\ 0.3 & 0.1 & 0.1 \end{bmatrix}$ and let's start out with the initial state vector $\vec{x}_0 = \begin{bmatrix} 1000 \\ 1600 \\ 4000 \end{bmatrix}$. We can find

$$\vec{x}_1 = \begin{bmatrix} 0.4 & 0 & 0.8 \\ 0.3 & 0.9 & 0.1 \\ 0.3 & 0.1 & 0.1 \end{bmatrix} \begin{bmatrix} 1000 \\ 1600 \\ 4000 \end{bmatrix} = \begin{bmatrix} 3600 \\ 2140 \\ 860 \end{bmatrix}$$

and

$$\vec{x}_2 = \begin{bmatrix} 0.4 & 0 & 0.8 \\ 0.3 & 0.9 & 0.1 \\ 0.3 & 0.1 & 0.1 \end{bmatrix} \begin{bmatrix} 3600 \\ 2140 \\ 860 \end{bmatrix} = \begin{bmatrix} 2128 \\ 3092 \\ 1380 \end{bmatrix}$$

for the population distributions for times $t = 1$ and $t = 2$. Notice that in each time step, the total population remains constant:

$$1000 + 1600 + 4000 = 3600 + 2140 + 860 = 2128 + 3092 + 1380 = 6600 \text{ people.}$$

If we want to find $\vec{x}_{10}$, we can compute

$$\vec{x}_{10} = A^{10}\vec{x}_0 = \begin{bmatrix} 0.4 & 0 & 0.8 \\ 0.3 & 0.9 & 0.1 \\ 0.3 & 0.1 & 0.1 \end{bmatrix}^{10} \begin{bmatrix} 100 \\ 1600 \\ 4000 \end{bmatrix} \approx \begin{pmatrix} 1233.890 \\ 4455.635 \\ 910.475 \end{pmatrix} \approx \begin{bmatrix} 1234 \\ 4456 \\ 910 \end{bmatrix}.$$

If we want to find $\lim_{t\to\infty} \vec{x}_t$, we will need to find the steady state vector. Since a steady state vector is an eigenvector for the eigenvalue of 1, we can find the nullspace of $A - I_3$. We set up the augmented matrix

$$\left[\begin{array}{ccc|c} -0.6 & 0 & 0.8 & 0 \\ 0.3 & -0.1 & 0.1 & 0 \\ 0.3 & 0.1 & -0.9 & 0 \end{array}\right]$$

and use row operations $r_1 \to 10r_1$, $r_2 \to 10r_2$ and $r_3 \to 10r_3$ to make it less horrible to look at:

$$\left[\begin{array}{ccc|c} -6 & 0 & 8 & 0 \\ 3 & -1 & 1 & 0 \\ 3 & 1 & -9 & 0 \end{array}\right].$$

The reduced row echelon form is

$$\left[\begin{array}{ccc|c} 1 & 0 & -4/3 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

and so we have

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = r \begin{bmatrix} 4/3 \\ 5 \\ 1 \end{bmatrix},$$

for some parameter r. Thus $\vec{x}_s = \begin{bmatrix} 4/3 \\ 5 \\ 1 \end{bmatrix}$ is a steady state vector. BUT! It is not going to be $\lim_{t\to\infty} \vec{x}_t$. That is because we want the steady state vector to also have the correct total population. Recall that the total population remains 6600 in our scenario. So we will have to adjust the parameter r to give us a steady state vector with a total population of 6600.

$$(4/3)r + 5r + r = 6600,$$

solving for r yields

$$r = 900.$$

So the steady state vector we want is

$$\vec{x}_s = 900 \begin{bmatrix} 4/3 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 1200 \\ 4500 \\ 900 \end{bmatrix}.$$

This steady state vector is the limit of $\vec{x}_t$ as $t \to \infty$. Notice that by the time we get

$$\vec{x}_{10} \approx \begin{bmatrix} 1234 \\ 4456 \\ 910 \end{bmatrix},$$

we are already fairly close to this limit.

The above is an example of a dynamical system. A dynamical system is any process wherein something's state is changed over time. They come in two flavors: discrete time and continuous time. We will primarily focus on discrete dynamical systems.

Example 23. Let's consider the dynamical system with variables x_t and y_t, and the relation

$$\begin{aligned} x_{t+1} &= 2.5x_t - 3y_t \\ y_{t+1} &= x_t - y_t. \end{aligned}$$

Then we can write the system as a matrix equation:

$$\begin{bmatrix} x_{t+1} \\ y_{t+1} \end{bmatrix} = \underbrace{\begin{bmatrix} 2.5 & -3 \\ 1 & -1 \end{bmatrix}}_{A} \begin{bmatrix} x_t \\ y_t \end{bmatrix}.$$

Note that A here is *not* a Markov matrix! So we can't necessarily take for granted that A has an eigenvector of 1 (it still could, but we would need to find that out ourselves).

Let's start with an initial state $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 70 \\ 30 \end{bmatrix}$. Then we can see:

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = A \begin{bmatrix} 70 \\ 30 \end{bmatrix} = \begin{bmatrix} 85 \\ 40 \end{bmatrix}$$
$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = A \begin{bmatrix} 85 \\ 40 \end{bmatrix} = \begin{bmatrix} 92.5 \\ 45 \end{bmatrix}$$
$$\begin{bmatrix} x_3 \\ y_3 \end{bmatrix} = A \begin{bmatrix} 92.5 \\ 45 \end{bmatrix} = \begin{bmatrix} 96.25 \\ 47.5 \end{bmatrix}.$$

It appears that the state vectors are getting closer and closer to $\begin{bmatrix} 100 \\ 50 \end{bmatrix}$. Can we confirm this?

One thing that we can do is to see if $\lim_{t\to\infty} A^t$ exists. If this limit does exist, then $\lim_{t\to\infty} \begin{bmatrix} x_t \\ y_t \end{bmatrix} = \left(\lim_{t\to\infty} A^t\right) \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$.

We can use the diagonalization to help us compute A^t. We will need to find the eigenvalues of A. We can take the characteristic polynomial

$$c_A(s) = \det \begin{bmatrix} 2.5 - s & -3 \\ 1 & -1 - s \end{bmatrix} = (2.5 - s)(-1 - s) + 3 = s^2 - 1.5s + 0.5.$$

We can use the quadratic formula to find that $c_A(s)$ has a root of $s = 1$ and a root of $s = 0.5$ Then we can write a diagonalization for A as

$$\underbrace{\begin{bmatrix} 2.5 & -3 \\ 1 & -1 \end{bmatrix}}_{A} = P \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}}_{D} P^{-1},$$

where we still have yet to find P. But what we have is already enough to know that $\lim_{t\to\infty} A^t$ exists! Because

$$\lim_{t\to\infty} A^t = \lim_{t\to\infty} \left(P \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix} P^{-1} \right)^t = \lim_{t\to\infty} P \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}^t P^{-1}$$
$$= \lim_{t\to\infty} P \begin{bmatrix} 1^t & 0 \\ 0 & 0.5^t \end{bmatrix} P^{-1} = P \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} P^{-1}.$$

So all we need to do is find P and plug it into the above formula. So we need to find eigenvectors for the eigenvalues 1 and 0.5.

To find an eigenvector for 1, we need a vector in the nullspace of $A - I_2$. We can set up the augmented matrix

$$\left[\begin{array}{cc|c} 1.5 & -3 & 0 \\ 1 & -2 & 0 \end{array}\right]$$

which has reduced row echelon form

$$\left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

Thus a parameterization for the solution set is

$$\begin{bmatrix} x \\ y \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \end{bmatrix},$$

and we can take $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ as our eigenvector. After doing a similar process to find the nullspace of $A - 0.5I_2$, we get $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ as an eigenvector for 0.5.

So now we can assemble

$$P = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix},$$

and so using the above formula,

$$\lim_{t\to\infty} A^t = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & -6 \\ 2 & -3 \end{bmatrix}.$$

Finally, we can use $\lim_{t\to\infty} A^t = \begin{bmatrix} 4 & -6 \\ 2 & -3 \end{bmatrix}$ to find the limit of $\begin{bmatrix} x_t \\ y_t \end{bmatrix}$ given the initial state vector $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 70 \\ 30 \end{bmatrix}$:

$$\lim_{t\to\infty} \begin{bmatrix} x_t \\ y_t \end{bmatrix} = \left(\lim_{t\to\infty} A^t\right) \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 4 & -6 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} 70 \\ 30 \end{bmatrix} = \begin{bmatrix} 100 \\ 50 \end{bmatrix},$$

which matches the initial observation!

Remark 2. Note here that as long as every eigenvalue of A is inside the interval $(-1, 1]$ (that is, the numbers x such that $-1 < x \leq 1$), then $\lim_{t\to\infty} A^t$ will exist. That is because only for numbers inside this interval will $\lim_{t\to\infty} x^t$ will exist. Specifically,

$$\lim_{t\to\infty} x^t = \begin{cases} 0 & \text{if } -1 < x < 1 \\ 1 & \text{if } x = 1 \end{cases}.$$

By contrast, $\lim_{t\to\infty}(-1)^t$ does not exist because the powers of -1 bounce back and forth between -1 and 1, and do not converge to anything. And if $x > 1$, then $\lim_{t\to\infty} x^t$ will diverge to ∞, and if $x < -1$, then $\lim_{t\to\infty} x^t$ will also diverge, flipping between positive and negative numbers of greater and greater absolute value.

CAUTION: Even if $\lim_{t\to\infty} A^t$ does not exist, the limit $\lim_{t\to\infty}(A^t\vec{v})$ CAN still exist, as long as $\vec{v}$ is a linear combination of eigenvectors with eigenvalues in the interval $(-1, 1]$.

For example, suppose A has three eigenvalues: 0.1, 1, and 10. Then $\lim_{t\to\infty} A^t$ does not exist, because A has an eigenvalue of 10, which is outside the required interval $(-1, 1]$. But since 0.1 is an eigenvalue, that means there is some vector $\vec{v}$ such that $A\vec{v} = 0.1\vec{v}$. That means that $A^2\vec{v} = 0.01\vec{v}$, $A^3\vec{v} = 0.001\vec{v}$, and so on. Taking the limit, we get

$$\lim_{t\to\infty}(A^t\vec{v}) = \lim_{t\to\infty} 0.1^t\vec{v} = \vec{0}.$$

Furthermore, since 1 is an eigenvalue, there is a vector $\vec{u}$ such that $A\vec{u} = \vec{u}$. And so $A^2\vec{u} = \vec{u}$, and $A^3\vec{u} = \vec{u}$. So

$$\lim_{t\to\infty}(A^t\vec{u}) = \vec{u}.$$

Finally, we can also do this for any linear combination of $\vec{v}$ and $\vec{u}$:

$$\lim_{t\to\infty}(A^t(a\vec{v} + b\vec{u})) = a\lim_{t\to\infty}(A^t\vec{v}) + b\lim_{t\to\infty}(A^t\vec{u}) = b\vec{u}.$$

One of the more interesting eigenvalues that can appear in a dynamical system is -1:

Example 24. Suppose we have the following dynamical system:

$$\begin{aligned} x_{t+1} &= -5.5x_t + 3y_t \\ y_{t+1} &= -9x_t + 5y_t \end{aligned}$$

. When we can write the system as a matrix equation:

$$\begin{bmatrix} x_{t+1} \\ y_{t+1} \end{bmatrix} = \underbrace{\begin{bmatrix} -5.5 & 3 \\ -9 & 5 \end{bmatrix}}_{A} \begin{bmatrix} x_t \\ y_t \end{bmatrix}.$$

Then let us consider the initial state vector $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 60 \\ 100 \end{bmatrix}$. Then we can see the states progress as follows:

$$\begin{aligned} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} &= A\begin{bmatrix} 60 \\ 100 \end{bmatrix} = \begin{bmatrix} -30 \\ -40 \end{bmatrix} \\ \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} &= A\begin{bmatrix} -30 \\ -40 \end{bmatrix} = \begin{bmatrix} 45 \\ 70 \end{bmatrix} \\ \begin{bmatrix} x_3 \\ y_3 \end{bmatrix} &= A\begin{bmatrix} 45 \\ 70 \end{bmatrix} = \begin{bmatrix} -37.5 \\ -55 \end{bmatrix} \\ \begin{bmatrix} x_4 \\ y_4 \end{bmatrix} &= A\begin{bmatrix} -37.5 \\ -55 \end{bmatrix} = \begin{bmatrix} 41.25 \\ 62.5 \end{bmatrix} \\ \begin{bmatrix} x_5 \\ y_5 \end{bmatrix} &= A\begin{bmatrix} 41.25 \\ 62.5 \end{bmatrix} = \begin{bmatrix} -39.375 \\ -58.75 \end{bmatrix}. \end{aligned}$$

These states certainly do not seem to be converging to anything. But even though it takes a bit longer to notice the pattern than last time, it seems the even-numbered states are converging to $\begin{bmatrix} 40 \\ 60 \end{bmatrix}$ and the odd-numbered states are converging to $\begin{bmatrix} -40 \\ -60 \end{bmatrix}$. Let's try to verify this.

We will begin by finding the eigenvalues of A. We have the characteristic polynomial

$$c_A(s) = \det\begin{bmatrix} -5.5 - s & 3 \\ -9 & 5 - s \end{bmatrix} = (-5.5 - s)(5 - s) + 27 = s^2 + 0.5s - 0.5.$$

Using the quadratic formula reveals that the eigenvalues are -1 and 0.5. Here we run into a problem: -1 is outside the required interval of $(-1, 1]$, so $\lim_{t\to\infty} A^t$ does not exist. To see why in more detail, we have a diagonalization for A:

$$\begin{bmatrix} -5.5 & 3 \\ -9 & 5 \end{bmatrix} = P \begin{bmatrix} -1 & 0 \\ 0 & 0.5 \end{bmatrix} P^{-1},$$

so

$$\begin{bmatrix} -5.5 & 3 \\ -9 & 5 \end{bmatrix}^t = P \begin{bmatrix} (-1)^t & 0 \\ 0 & 0.5^t \end{bmatrix} P^{-1},$$

but $\lim_{t\to\infty} (-1)^t$ does not exist. But there is a way to rescue the situation! Even though $\lim_{t\to\infty} (-1)^t$ does not exist, we can break this sequence up into two branches: an even branch and an odd branch. We have

$$\lim_{t\to\infty} (-1)^{2t} = 1$$
$$\lim_{t\to\infty} (-1)^{2t+1} = -1$$

because the even powers of -1 are all 1 (so that sequence converges to 1) and all the odd powers of -1 are -1, so the odd sequence converges to -1.

So even though $\lim_{t\to\infty} A^t$ does not exist, the limits $\lim_{t\to\infty} A^{2t}$ and $\lim_{t\to\infty} A^{2t+1}$ both exist, and we can find those:

$$\lim_{t\to\infty} A^{2t} = \lim_{t\to\infty} P \begin{bmatrix} (-1)^{2t} & 0 \\ 0 & 0.5^{2t} \end{bmatrix} P^{-1} = P \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} P^{-1},$$
$$\lim_{t\to\infty} A^{2t+1} = \lim_{t\to\infty} P \begin{bmatrix} (-1)^{2t+1} & 0 \\ 0 & 0.5^{2t} \end{bmatrix} P^{-1} = P \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} P^{-1}.$$

Now all we need is to find P and we will have our limits. We need to find eigenvectors for the eigenvalues of -1 and 0.5. So we need to compute the nullspaces of $A + I_2$ and $A - 0.5I_2$. For $A + I_2$, we get the augmented matrix

$$\left[\begin{array}{cc|c} -4.5 & 3 & 0 \\ -9 & 6 & 0 \end{array}\right]$$

which has reduced row echelon form

$$\left[\begin{array}{cc|c} 1 & -2/3 & 0 \\ 0 & 0 & 0 \end{array}\right],$$

so a parameterization for the solution set is

$$\begin{bmatrix} x \\ y \end{bmatrix} = t \begin{bmatrix} 2/3 \\ 1 \end{bmatrix}.$$

Choosing $t = 3$ allows us to pick an eigenvector without any fractions: $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$. A similar process for $A - 0.5I_2$ yields an eigenvector of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ for the eigenvalue 0.5. Thus we can construct P as

$$P = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$$

and so using the formulae above:

$$\lim_{t\to\infty} A^{2t} = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 6 & -3 \end{bmatrix},$$

$$\lim_{t\to\infty} A^{2t+1} = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ -6 & 3 \end{bmatrix}.$$

Now we can compute the limits of the even-numbered subsequence and the odd-numbered subsequence of $\begin{bmatrix} x_t \\ y_t \end{bmatrix}$, as follows

$$\lim_{t\to\infty} \begin{bmatrix} x_{2t} \\ y_{2t} \end{bmatrix} = \left(\lim_{t\to\infty} A^{2t}\right) \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 6 & -3 \end{bmatrix} \begin{bmatrix} 60 \\ 100 \end{bmatrix} = \begin{bmatrix} 40 \\ 60 \end{bmatrix}$$

and

$$\lim_{t\to\infty} \begin{bmatrix} x_{2t+1} \\ y_{2t+1} \end{bmatrix} = \left(\lim_{t\to\infty} A^{2t+1}\right) \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ -6 & 3 \end{bmatrix} \begin{bmatrix} 60 \\ 100 \end{bmatrix} = \begin{bmatrix} -40 \\ -60 \end{bmatrix},$$

which matches the observation from above!

Note that in this example, A has no steady state vector because 1 is not an eigenvalue of A. But instead, notice that the eigenvector $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ for -1 exhibits a behavior *similar* to a steady state vector: $A\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \end{bmatrix}$ and $A\begin{bmatrix} -2 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, so an initial state of $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ will end up bouncing back and forth between two states forever. This makes $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ a **periodic** state, with a cycle length of 2, because it takes the vector two turns to get back where it started.

A natural question is to then ask: Can there be periodic states with a higher cycle length? And the answer is yes! But to understand them we will first have to learn about complex numbers, coming up after the midterm!