Enumeration Puzzles in Geometry

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Recap

In Frank's talk, we learned about the Chow ring $A(X)$ of a smooth variety X. The Chow ring is $Z(X)/\text{Rat}(X)$: the cycles of X modulo rational equivalence.

What is an enumeration puzzle?

Enumeration puzzles in algebraic geometry ask us to describe the set Φ of objects of a certain type satisfying a number of conditions. In the most common situation, we expect Φ to be finite and we ask for its cardinality, whence the name enumerative geometry.

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E.G.: How many lines go through two points in the plane? How many lines in $\mathbb{P}^3$ meet each of four given lines? How many conics are tangent to five given conics in the plane?

(Answers: 1, 2, 3264)

Steps to an enumeration puzzle

1. Find a parameter space H for the objects we seek.
2. Describe the Chow ring $A(H)$.
3. Find the classes $[Z_i] \in A(H)$ of the loci of objects satisfying the conditions imposed.
4. Calculate the product $\alpha = \prod [Z_i]$.
5. Verify that the cycle corresponding to α has correct dimension, and investigate its geometry.

Getting started

Let's focus on our guiding question: given four lines $L_1, \ldots, L_4 \subseteq \mathbb{P}^3_k$ (k algebraically closed, char $k = 0$), how many lines in $\mathbb{P}^3_k$ intersect all four?

We will be following Chapter 3 of Eisenbud and Harris: *3264 & All That Intersection Theory in Algebraic Geometry*.

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First, let's come up with H. For that, we will need the help of Grassmannians.

Grassmannians

A Grassmannian is a projective variety whose points correspond to vector subspaces of a certain dimension of a given vector space.

For example: $G(1,3)$ is the space of 1-dimensional vector subspaces of k^3. Since every line through the origin of k^3 corresponds to a point in $\mathbb{P}^2$, we can say $G(1,3) = \mathbb{P}^2 = PG(0,2)$.

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Another: $G(2,3)$ is the space of 2-dimensional vector subspaces of k^3. Every plane through the origin of k^3 corresponds to a line in $\mathbb{P}^2$, so $G(2,3) = PG(1,2)$.

$f : PG(1,2) \to \mathbb{P}^2$ given by $f(V(ax+by+cz)) = (a,b,c)$ is an isomorphism.

Step 1: Getting to know $H = PG(1,3)$

We are interested in lines in $\mathbb{P}^3_k$:

$$PG(1,3) = G(2,4) \overset{\mathcal{P}}{\hookrightarrow} \mathbb{P}\left(\bigwedge^2 k^4\right) = \mathbb{P}^5_k,$$

where $\bigwedge^2 k^4$ is a $\binom{4}{2} = 6$-dimensional k-vector space generated by $\{b_i \wedge b_j : b_i, b_j \in \text{Basis}(k^4), i < j\}$.

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$\mathcal{P}$ is the *Plücker embedding*: $\mathcal{P}(\langle u, v\rangle) = [u \wedge v]$. This is well-defined and injective.

Waiter, there's a wedge in my product

How can we tell which elements of $\bigwedge^2 k^4$ are in the image of $G(2,4)$? The ones that can be rewritten into simple products!

$$b_1 \wedge b_2 + b_1 \wedge b_3 = b_1 \wedge (b_2 + b_3)$$

but

$$b_1 \wedge b_2 + b_3 \wedge b_4$$

is stuck.

How can we tell whether

$$\sum_{1 \leq i < j \leq 4} p_{i,j} b_i \wedge b_j$$

can be rewritten this way?

The Klein Quadric

$$\left(\sum_{1 \le i < j \le 4} p_{i,j} b_i \wedge b_j \right)^{\wedge 2} = 0$$

if and only if

$$p_{1,2}p_{3,4} - p_{1,3}p_{2,4} + p_{1,4}p_{2,3} = 0.$$

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So

$$PG(1,3) \cong V(p_{1,2}p_{3,4} - p_{1,3}p_{2,4} + p_{1,4}p_{2,3}) \subseteq \mathbb{P}^5_k$$

is a smooth quadric hypersurface (the *Klein quadric*).

Step 2: The Chow ring $A(PG(1,3))$

To begin, we will need to fix an arbitrary *complete flag* $\mathcal{V}$ of $\mathbb{P}^3$:

$$\mathcal{V} = (p \in L \subseteq \Pi \subseteq \mathbb{P}^3).$$

We can define the following subvarieties called *Schubert cycles*.

$$\begin{aligned}
\Sigma_{0,0}(\mathcal{V}) &= PG(1,3) \\
\Sigma_{1,0}(\mathcal{V}) &= \{\Lambda : \Lambda \cap L \neq \varnothing\} \\
\Sigma_{2,0}(\mathcal{V}) &= \{\Lambda : p \in \Lambda\} \\
\Sigma_{1,1}(\mathcal{V}) &= \{\Lambda : \Lambda \subseteq \Pi\} \\
\Sigma_{2,1}(\mathcal{V}) &= \{\Lambda : p \in \Lambda \subseteq \Pi\} \\
\Sigma_{2,2}(\mathcal{V}) &= \{\Lambda : \Lambda = L\}
\end{aligned}$$

Schubert Classes

The Schubert cycles of the same type are rationally equivalent regardless of flag. That is,

$$\Sigma_{a,b}(\mathcal{V}) \simeq \Sigma_{a,b}(\mathcal{V}').$$

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The fact that they generate A comes from the fact that the Schubert cells $\Sigma^{\circ}_{a,b}$ form an *affine stratification*.

Ring Structure

The Schubert classes follow the following multiplicative structure:

$$
\begin{aligned}
\sigma_{1,0}^2 &= \sigma_{1,1} + \sigma_{2,0} \\
\sigma_{1,0}\sigma_{1,1} = \sigma_{1,0}\sigma_{2,0} &= \sigma_{2,1} \\
\sigma_{1,2}\sigma_{2,1} &= \sigma_{2,2} \\
\sigma_{1,1}^2 = \sigma_{2,0}^2 &= \sigma_{2,2} \\
\sigma_{1,1}\sigma_{2,0} &= 0
\end{aligned}
$$

We will demonstrate this structure using intersection theory!

$\sigma_{1,1}^2 = \sigma_{2,0}^2 = \sigma_{2,2}$

Take two different flags: $\mathcal{V} = (p \in L \subseteq \Pi)$ and $\mathcal{V}' = (p' \in L' \subseteq \Pi')$.

$$\sigma_{1,1}^2 = \#(\Sigma_{1,1}(\mathcal{V}) \cap \Sigma_{1,1}(\mathcal{V}')) \cdot \sigma_{2,2} = \#\{\Pi \cap \Pi'\} \cdot \sigma_{2,2} = \sigma_{2,2}.$$

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$$\sigma_{2,0}^2 = \#(\Sigma_{2,0}(\mathcal{V}) \cap \Sigma_{2,0}(\mathcal{V}')) \cdot \sigma_{2,2} = \#\{\overline{pp'}\} \cdot \sigma_{2,2} = \sigma_{2,2}.$$

$\sigma_{1,1}\sigma_{2,0} = 0$

$$\sigma_{1,1}\sigma_{2,0} = \#(\Sigma_{1,1}(\mathcal{V}) \cap \Sigma_{2,0}(\mathcal{V}')) \cdot \sigma_{2,2}.$$

But $\Sigma_{1,1}(\mathcal{V}) \cap \Sigma_{2,0}(\mathcal{V}')$ comprises the lines that both contain p' and are contained in Π. Such a line can only exist if $p' \in \Pi$, which does not hold in general. Therefore $\#(\Sigma_{1,1}(\mathcal{V}) \cap \Sigma_{2,0}(\mathcal{V}')) = 0$ and so $\sigma_{1,1}\sigma_{2,0} = 0$.

$\sigma_{1,0}\sigma_{2,1} = \sigma_{2,2}$

$$\sigma_{1,0}\sigma_{2,1} = \#(\Sigma_{1,0}(\mathcal{V}) \cap \Sigma_{2,1}(\mathcal{V}')) \cdot \sigma_{2,2}.$$

Now

$$\Sigma_{1,0}(\mathcal{V}) \cap \Sigma_{2,1}(\mathcal{V}') = \{\Lambda : \Lambda \cap L \neq \varnothing, p' \in \Lambda \subseteq \Pi'\}.$$

Let $q = L \cap \Pi'$. The only choice for Λ is $\overline{qp'}$.

Therefore $\#(\Sigma_{1,0}(\mathcal{V}) \cap \Sigma_{2,1}(\mathcal{V}')) = 1$ and so $\sigma_{1,0}\sigma_{2,1} = \sigma_{2,2}$.

$\sigma_{1,0}\sigma_{2,0} = \sigma_{1,0}\sigma_{1,1} = \sigma_{2,1}$

$$\begin{aligned}\Sigma_{1,0}(\mathcal{V}) \cap \Sigma_{2,0}(\mathcal{V}') &= \{\Lambda : \Lambda \cap L \neq \varnothing, p' \in \Lambda\} \\ &= \{\Lambda : p' \in \Lambda \subseteq \overline{p'L}\} = \Sigma_{2,1}(p' \in \widetilde{L} \subseteq \overline{p'L})\end{aligned}$$

where $\widetilde{L}$ is any line containing p' in $\overline{p'L}$.

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$$= \{\Lambda : (L \cap \Pi') \in \Lambda \subseteq \Pi'\} = \Sigma_{2,1}((L \cap \Pi') \in \widetilde{L}' \subseteq \Pi'\}$$

where $\widetilde{L}'$ is any line containing $L \cap \Pi'$ in Π'.

$\sigma_{1,0}^2 = \sigma_{1,1} + \sigma_{2,0}$

$$\Sigma_{1,0}(\mathcal{V}) \cap \Sigma_{1,0}(\mathcal{V}') = \{\Lambda : \Lambda \cap L \neq \varnothing, \Lambda \cap L' \neq \varnothing\}.$$

This is not a Schubert cycle of any flag!

New idea: $\sigma_{1,0}^2 = \alpha\sigma_{1,1} + \beta\sigma_{2,0}$, solve.

$$(\alpha\sigma_{1,1} + \beta\sigma_{2,0})\sigma_{1,1} = \sigma_{1,0}^2\sigma_{1,1} = \sigma_{1,0}\sigma_{2,1} = \sigma_{2,2}.$$

Also

$$(\alpha\sigma_{1,1} + \beta\sigma_{2,0})\sigma_{1,1} = \alpha\sigma_{1,1}^2 + 0\beta = \alpha\sigma_{2,2}$$

So $\alpha = 1$. Similarly, $\beta = 1$.

How many lines meet four given lines?

Suppose L_1, L_2, L_3, L_4 are general lines in $\mathbb{P}^3$. First let us device four distinct flags $\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3, \mathcal{V}_4$.

Translate the question: what is the cardinality of $\bigcap_{i=1}^4 \Sigma_{1,0}(\mathcal{V}_i)$?

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That is the degree of the class $\sigma_{1,0}^4 \in A(PG(1,3))$.

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Translate the question: what is the cardinality of $\bigcap_{i=1}^4 \Sigma_{1,0}(\mathcal{V}_i)$?

That is the degree of the class $\sigma_{1,0}^4 \in A(PG(1,3))$.

$$\sigma_{1,0}^4 = (\sigma_{1,1} + \sigma_{2,0})^2 = \sigma_{2,2} + 2 \cdot 0 + \sigma_{2,2} = 2\sigma_{2,2}.$$

Since $\sigma_{2,2}$ is the class of $\Sigma_{2,2} = \{\texttt{line}\}$, $2\sigma_{2,2}$ gives us the answer of two lines!

Why study enumeration puzzles?

Not only are these puzzles interesting in their own right, but they are also a great way to learn some of the more advanced concepts in algebraic geometry!

We've already made use of Grassmannians and intersection theory to uncover this Chow ring. In more advanced puzzles, we may need more tools to uncover the correct parameter space and to deal with "excess intersection."