# 1 Motivation

When studying dynamical systems, we encountered **steady-state vectors** and **periodic state vectors**. Given a matrix $A$, a steady-state vector for $A$ is an eigenvector for the eigenvalue 1. That is, $\vec{v}$ is a steady-state vector if $A\vec{v} = \vec{v}$. A vector $\vec{u}$ is periodic if there is some $t \geq 1$ such that $A^t\vec{u} = \vec{u}$. The smallest positive $t$ such that $A^t\vec{u} = \vec{u}$ is called the period (or order) of $\vec{u}$.

For example, $\begin{bmatrix} 2.5 & -3 \\ 1 & -1 \end{bmatrix}$ has eigenvalues 1 and 0.5. The vector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector for the eigenvalue 1, so $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is a steady-state vector. We can verify:

$$\begin{bmatrix} 2.5 & -3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5-3 \\ 2-1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

(Note: steady-state vectors are periodic vectors whose period is $t = 1$.)

The matrix $\begin{bmatrix} -5.5 & 3 \\ -9 & 5 \end{bmatrix}$ has eigenvalues $-1$ and 0.5. The vector $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ is an eigenvector for $-1$. Therefore

$$\begin{bmatrix} -5.5 & 3 \\ -9 & 5 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -11+9 \\ -18+15 \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \end{bmatrix}.$$

Then we can multiply again and get

$$\begin{bmatrix} -5.5 & 3 \\ -9 & 5 \end{bmatrix} \begin{bmatrix} -2 \\ -3 \end{bmatrix} = \begin{bmatrix} 11-9 \\ 18-15 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

So

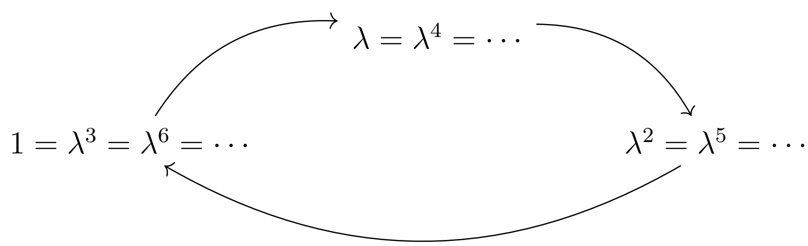
$$\begin{bmatrix} -5.5 & 3 \\ -9 & 5 \end{bmatrix}^2 \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix},$$

so $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ is periodic with period $t = 2$.

Now imagine a matrix $A$ with a periodic vector $\vec{v}$ whose period is $t = 3$. For example, in the system

$$A = \begin{bmatrix} 1 & 1 \\ -3 & -2 \end{bmatrix},$$

every vector is periodic of order $t = 3$. That means $A\vec{v} \neq \vec{v}$, $A^2\vec{v} \neq \vec{v}$, but $A^3\vec{v} = \vec{v}$. That means $\vec{v}$ is a steady-state of $A^3$, so $A^3$ has an eigenvalue of 1. Then $A$ has some eigenvalue $\lambda$ whose cube is 1, but $\lambda \neq 1$. So the powers of $\lambda$ form a three-long cycle.




There is no number in $\mathbb{R}$ that has this property.

# 2 The Complex Numbers and Cartesian Form

We will begin by introducing the imaginary unit $i$. The imaginary unit satisfies the equation

$$i^2 = -1.$$

From this we will be able to construct the entire complex plane.

**Definition 1.** The **complex plane** is the set

$$\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}.$$

That is, $\mathbb{C}$ is the set of all numbers of the form $a + bi$ where $a$ and $b$ are any real number.

For example, $\mathbb{C}$ contains the numbers $1 + i$, $2 + 7i$, $3 - 16i$, $-9$, $17i$, etc. Note that by setting $b = 0$, $a + bi$ is just $a$, which is a real number. That means that the real numbers are contained in the complex numbers, or $\mathbb{R} \subseteq \mathbb{C}$.

Given two complex numbers $a + bi$ and $c + di$, we can add:

$$(a + bi) + (c + di) = a + c + bi + di = (a + c) + (b + d)i$$

and we can multiply:

$$(a + bi)(c + di) = ac + bic + adi + bdi^2 = ac + bdi^2 + (ad + bc)i = (ac - bd) + (ad + bc)i.$$

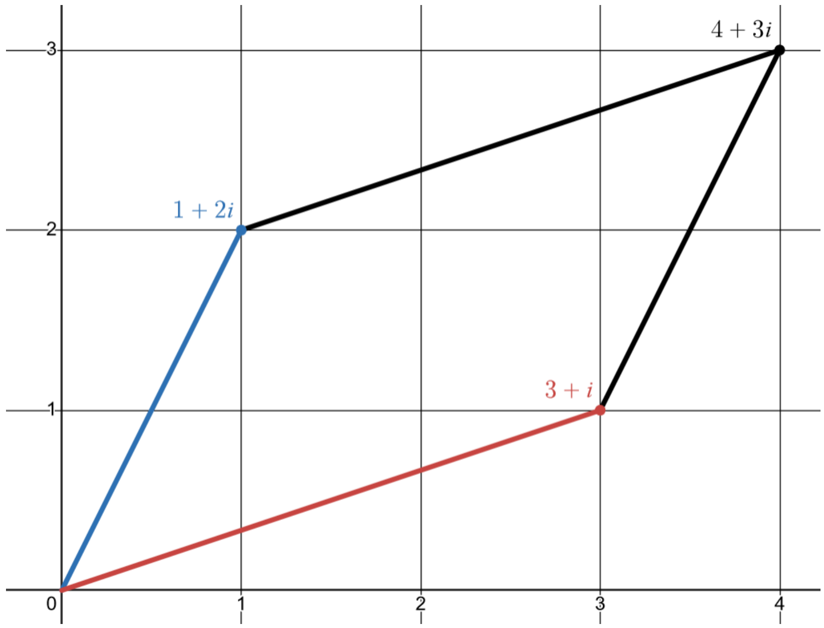
The highlighted expressions are in *Cartesian form*.

**Definition 2.** A complex number is in **Cartesian form** if it is written in the form $a + bi$.

That is, the expression $(1+i)+(2-3i)$ is not in Cartesian form, but $3-2i$ is. The expression $(4 + 5i)(9 - i)$ is not in Cartesian form, but $41 + 41i$ is.

**Definition 3.** The complex number $a + bi$ has a **real component** $a$ and an **imaginary component** $b$.

Complex numbers can be represented geometrically as points in a plane whose horizontal axis represents the real component and whose vertical axis represents the imaginary component. Then adding complex numbers is exactly the same as adding vectors in $\mathbb{R}^2$, as illustrated below.




# 3 Euler's Formula and Polar Form

We will begin by introducing Euler's formula:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

where $\theta$ is any number, representing an angle in the complex plane in radians, and $e$ is Euler's constant which you may recall from calculus.

We can see where Euler's formula comes from using Taylor series:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots,$$

so

$$e^{i\theta} = 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} + \cdots$$

which can be broken into Taylor series for cosine and sine:

$$\underbrace{\left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right)}_{\cos(\theta)} + i\underbrace{\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)}_{\sin(\theta)}.$$

**Definition 4.** A complex number is in **Polar form** if it is written in the form $re^{\theta i}$ for some real number $\theta$ (called the **angle** or **argument**) and some real number $r \geq 0$ (called the **radius** or **absolute value**).

The radius of a complex number $a + bi$ is $r = \sqrt{a^2 + b^2}$ (using the Pythagorean Theorem). The angle of $a + bi$ requires a little bit of trig to find: $\theta$ is either $\arctan(b/a)$ if $a > 0$, $\arctan(b/a) + \pi$ if $a < 0$, $\pi/2$ if $a = 0$ and $b > 0$, or $3\pi/2$ if $a = 0$ and $b < 0$. Note that if

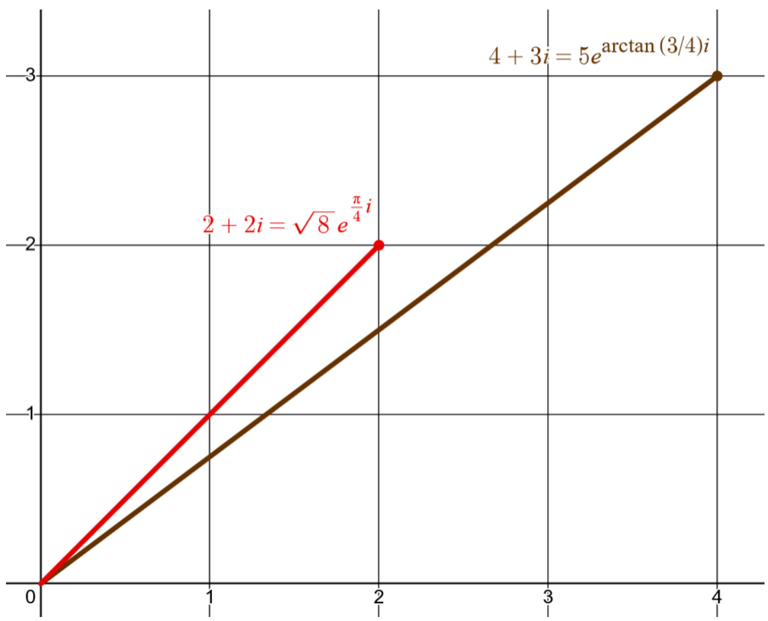
$a = 0$ and $b = 0$, then $a + bi = 0$ and the number 0 does not really have an angle, and so its polar form is just 0.

Below are formulas to convert between polar and Cartesian forms:

$$\underbrace{re^{\theta i} = r\cos(\theta) + ri\sin(\theta)}_{\text{Polar to Cartesian}},$$

$$\underbrace{a + bi = \begin{cases} \sqrt{a^2+b^2}e^{i\arctan(b/a)} & \text{if } a > 0 \\ \sqrt{a^2+b^2}e^{i(\arctan(b/a)+\pi)} & \text{if } a < 0 \\ \sqrt{a^2+b^2}e^{\frac{\pi i}{2}} & \text{if } a = 0 \text{ and } b > 0 \\ \sqrt{a^2+b^2}e^{\frac{3\pi i}{2}} & \text{if } a = 0 \text{ and } b < 0 \\ 0 & \text{if } a = b = 0 \end{cases}}_{\text{Cartesian to Polar}}.$$

Below is a geometric representation of this conversion.




Given two numbers in polar form $r_1e^{\theta_1 i}$ and $r_2e^{\theta_2 i}$, we can multiply:

$$\left(r_1e^{\theta_1 i}\right)\left(r_2e^{\theta_2 i}\right) = r_1r_2e^{\theta_1 i}e^{\theta_2 i} = r_1r_2e^{(\theta_1+\theta_2)i},$$

we can divide:

$$\left(r_1e^{\theta_1 i}\right) / \left(r_2e^{\theta_2 i}\right) = \frac{r_1}{r_2}e^{(\theta_1-\theta_2)i},$$

and we can raise $re^{\theta i}$ to the power of $t$:

$$\left(re^{\theta i}\right)^t = r^te^{\theta ti}.$$

That is, multiplying two complex numbers ends up **multiplying their radii** and **adding their angles**.

Dividing two complex numbers ends up **dividing their radii** and **subtracting their angles**.

Raising a complex number to the power of $t$ ends up **raising the radius to the power of $t$** and **multiplying the angle by $t$**.

For example,

$$\left(3e^{\frac{\pi i}{6}}\right)\left(4e^{\frac{\pi i}{3}}\right) = 12e^{\frac{\pi i}{2}} = 12\cos(\pi/2) + 12i\sin(\pi/2) = 12i.$$

Another example,

$$\left(e^{\frac{\pi i}{4}}\right)^2 = e^{\frac{\pi i}{2}} = \cos(\pi/2) + i\sin(\pi/2) = i.$$

So $e^{\frac{\pi i}{4}}$ turns out to be a *square root* of $i$. In fact, we can use polar form to find any $n^{\text{th}}$ root by using $t = 1/n$: for example

$$i^{1/3} = \left(e^{\frac{\pi i}{2}}\right)^{1/3} = e^{\frac{\pi i}{6}}.$$

Of course, just like with the real numbers, complex numbers can have more than one $n^{\text{th}}$ root. For example, $-e^{\frac{\pi i}{4}}$ is another square root of $i$. And $e^{\frac{5\pi i}{6}}$ and $-i$ are the other cube roots of $i$. In fact,

**Theorem 1.** Let $z \neq 0$ be a complex number. Then $z$ the number of $n^{\text{th}}$ roots of $z$ is $n$.

To be able to find all $n$ $n^{\text{th}}$ roots of $z$, we need to first understand all the $n^{\text{th}}$ roots of 1, called the roots of unity.

# 4 Roots of Unity

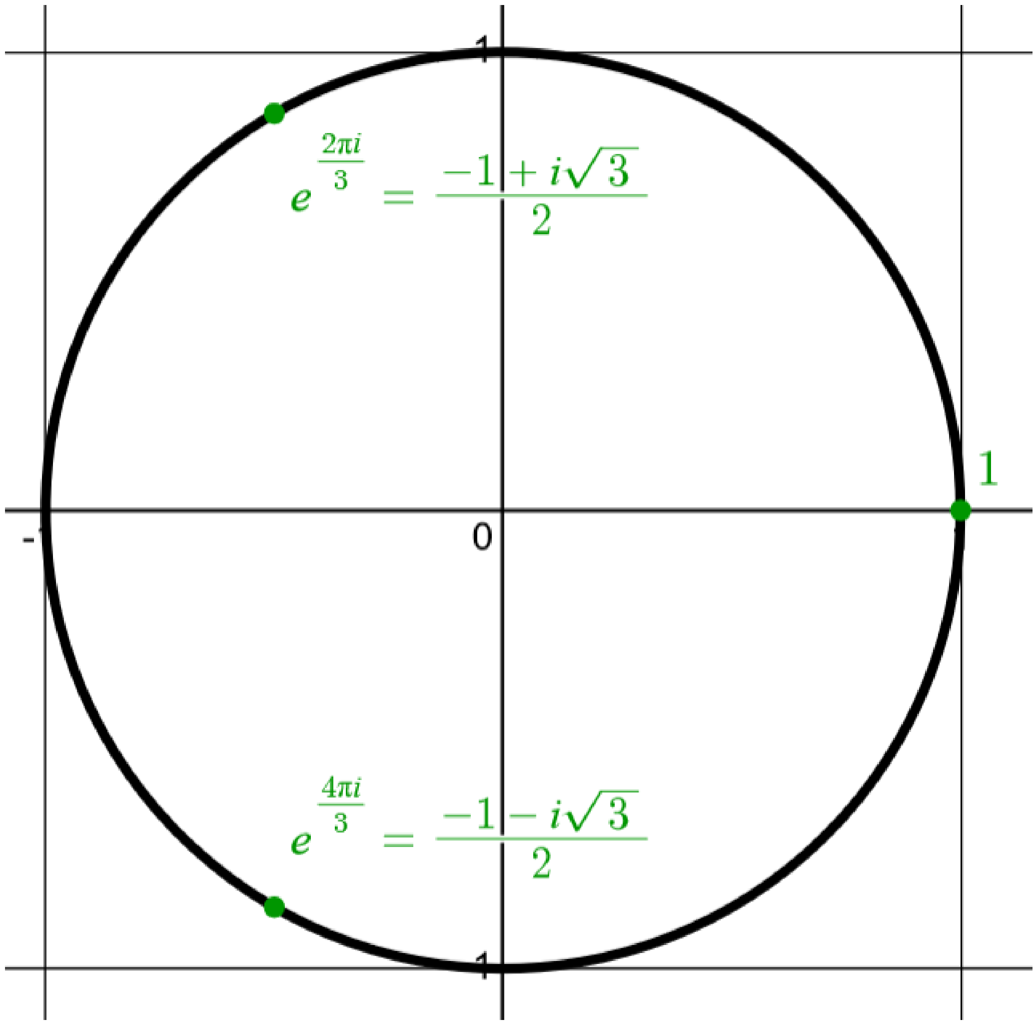
**Definition 5.** Given some positive integer $n$, an $n^{\text{th}}$ **root of unity** is a complex number $z$ such that

$$z^n = 1.$$

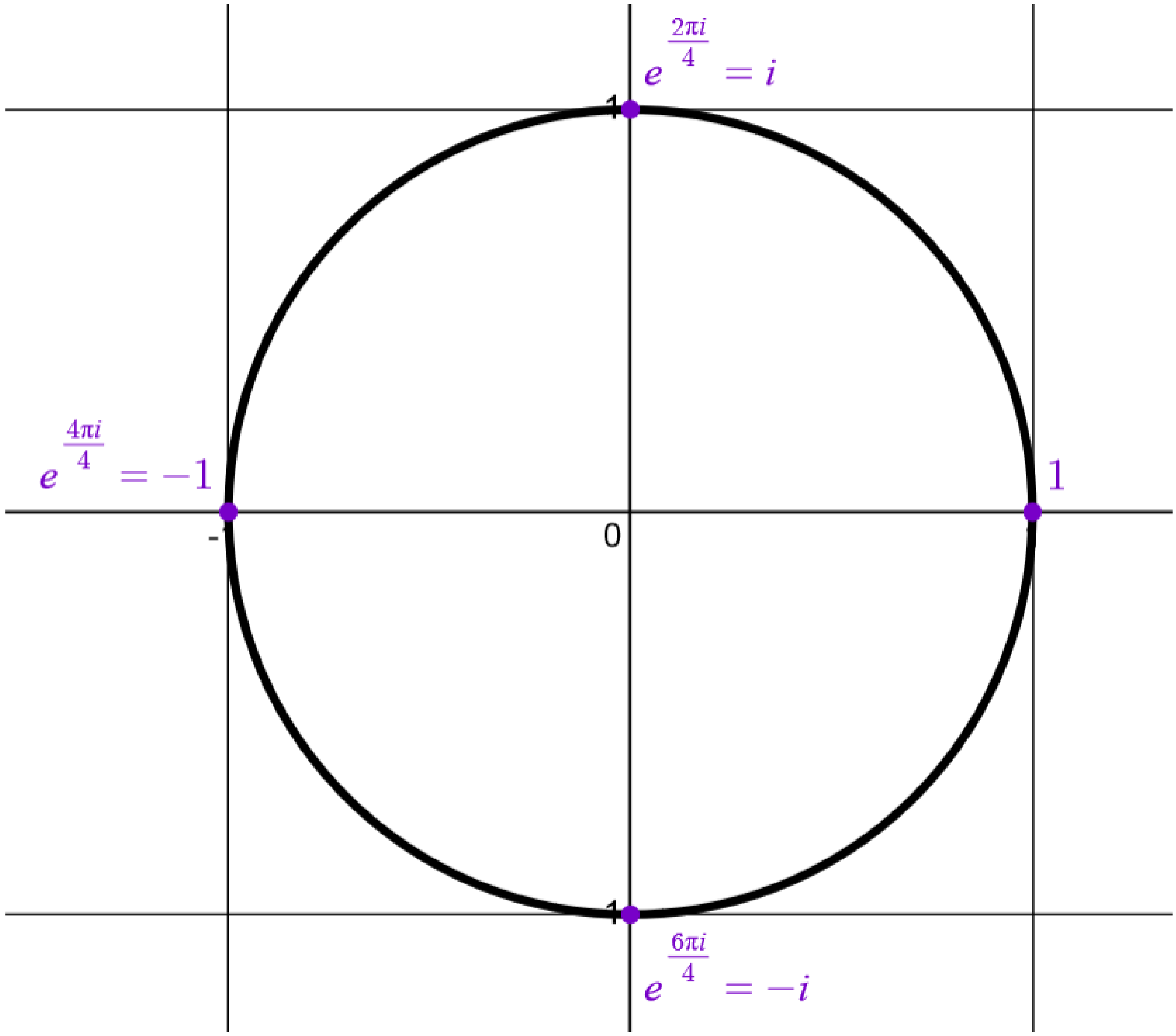
All roots of unity have a radius of 1 (that is, they are on the **unit circle**). We can use polar coordinates to express the $n^{\text{th}}$ roots of unity in the following table.

| $n$ | The $n^{\text{th}}$ roots of unity |
|---|---|
| 1 | 1 |
| 2 | $1, -1$ |
| 3 | $1, e^{\frac{2\pi i}{3}}, e^{\frac{4\pi i}{3}}$ |
| 4 | $1, e^{\frac{2\pi i}{4}}, e^{\frac{4\pi i}{4}}, e^{\frac{6\pi i}{4}}$ |
| 5 | $1, e^{\frac{2\pi i}{5}}, e^{\frac{4\pi i}{5}}, e^{\frac{6\pi i}{5}}, e^{\frac{8\pi i}{5}}$ |
| $\vdots$ | $\vdots$ |

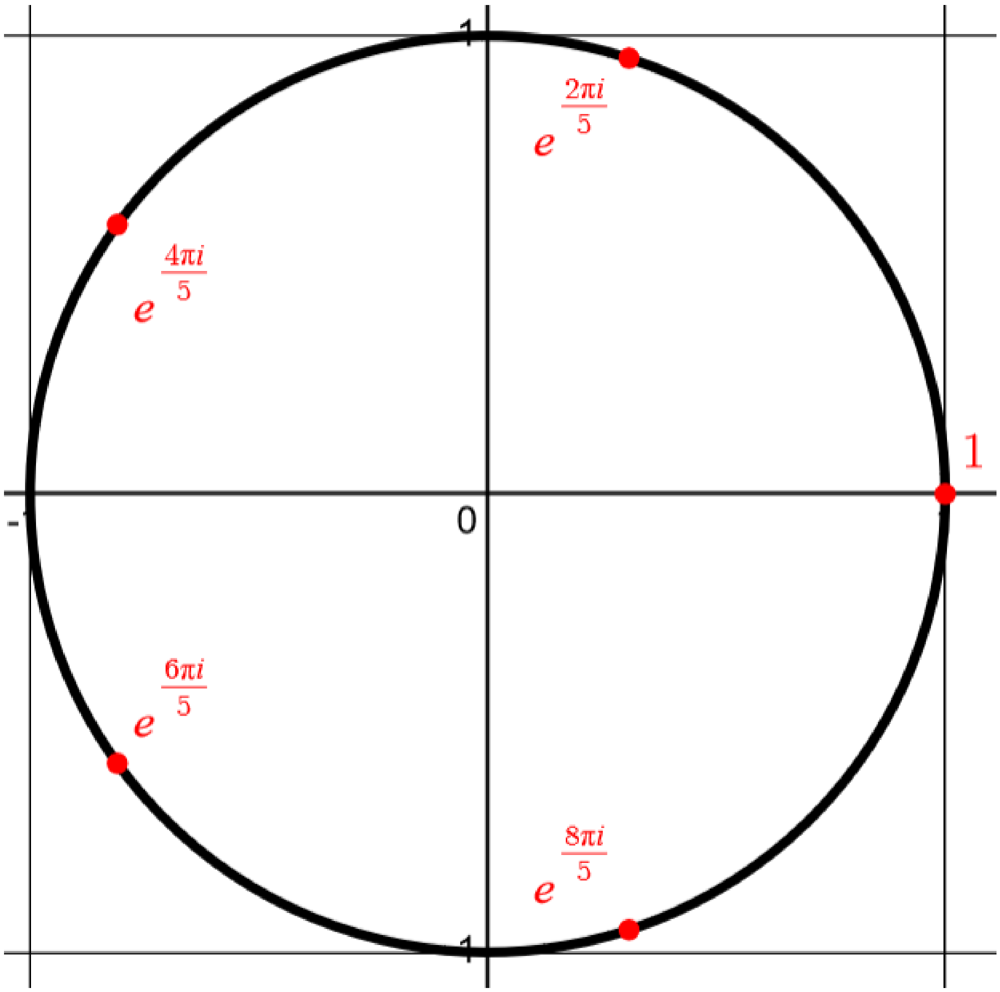
Below are some graphs of the 3rd, 4th, 5th, and 6th roots of unity, with the unit circle drawn in black.



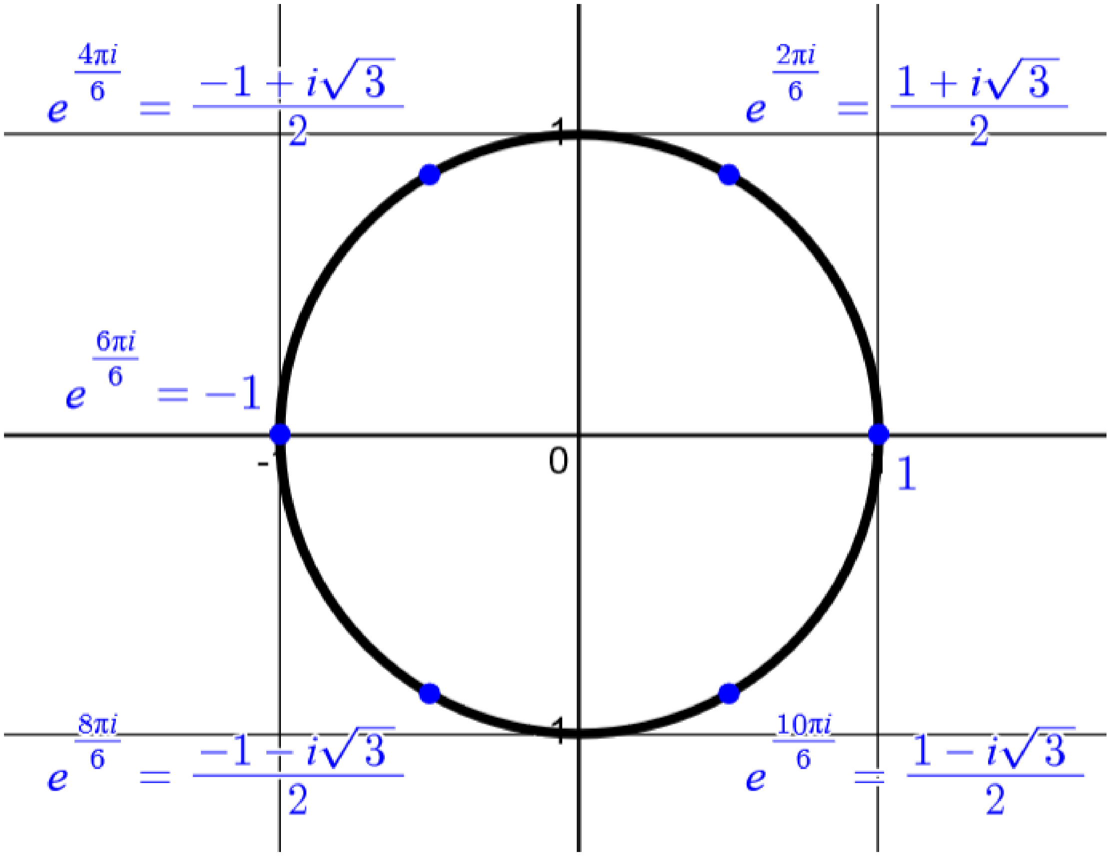

**The third roots of unity divide the unit circle into three equal parts.**




**The fourth roots of unity divide the unit circle into four equal parts.**




**The fifth roots of unity divide the unit circle into five equal parts.**




**The sixth roots of unity divide the unit circle into six equal parts.**

Some people use the Greek letter zeta to denote the $n^{\text{th}}$ root of unity with the smallest positive angle: $\zeta_n = e^{\frac{2\pi i}{n}}$. But omega is also commonly used: $\omega_n = e^{\frac{2\pi i}{n}}$.

**Theorem 2.** Let $z$ be any nonzero complex number and let $r$ be one of the $n^{\text{th}}$ roots of $z$. Then

$$r\left(e^{\frac{2\pi i}{n}}\right)^t$$

is another $n^{\text{th}}$ root of $z$. By using any number between 0 and $n-1$ for $t$, we end up using all $n$ $n^{\text{th}}$ roots of unity.

For example, take $z = 8$ and let's find the three cube roots. We know that one of the cube roots of 8 is 2. To find the other two, we can take the number 2 and multiply it by the third roots of unity to get

$$2,\ 2e^{\frac{2\pi i}{3}},\ 2e^{\frac{4\pi i}{3}}$$

as all three cube roots of 8.

For another example, take $z = -81$ and let's find the four fourth roots. Now $-81$ does not have any real fourth roots, so it will help to convert to polar coordinates to find one. The number $-81$ has a radius of 81 and an angle of $\pi$, so we can write

$$-81 = 81e^{\pi i}.$$

Now we can take a fourth root by raising this to the power of 1/4:

$$(-81)^{\frac{1}{4}} = \left(81e^{\pi i}\right)^{\frac{1}{4}} = 81^{\frac{1}{4}}e^{\frac{\pi i}{4}} = 3e^{\frac{\pi i}{4}}.$$

Therefore $3e^{\frac{\pi i}{4}}$ is one of the fourth roots of $-81$. To find the other three, we can take the root we just found $3e^{\frac{\pi i}{4}}$, and multiply it by the fourth roots of unity 1, $e^{\frac{2\pi i}{4}}$, $e^{\frac{4\pi i}{4}}$, and $e^{\frac{6\pi i}{4}}$. We end up getting

$$3e^{\frac{\pi i}{4}},\ 3e^{\frac{3\pi i}{4}},\ 3e^{\frac{5\pi i}{4}},\ 3e^{\frac{7\pi i}{4}}$$

as all four fourth roots of $-81$ (after simplifying the polar forms). If we want, we can also we can also express these answers in Cartesian coordinates as follows:

$$\left(\frac{3\sqrt{2}}{2} + i\frac{3\sqrt{2}}{2}\right),\ \left(-\frac{3\sqrt{2}}{2} + i\frac{3\sqrt{2}}{2}\right),\ \left(-\frac{3\sqrt{2}}{2} - i\frac{3\sqrt{2}}{2}\right),\ \left(\frac{3\sqrt{2}}{2} - i\frac{3\sqrt{2}}{2}\right).$$

# 5 Conjugation

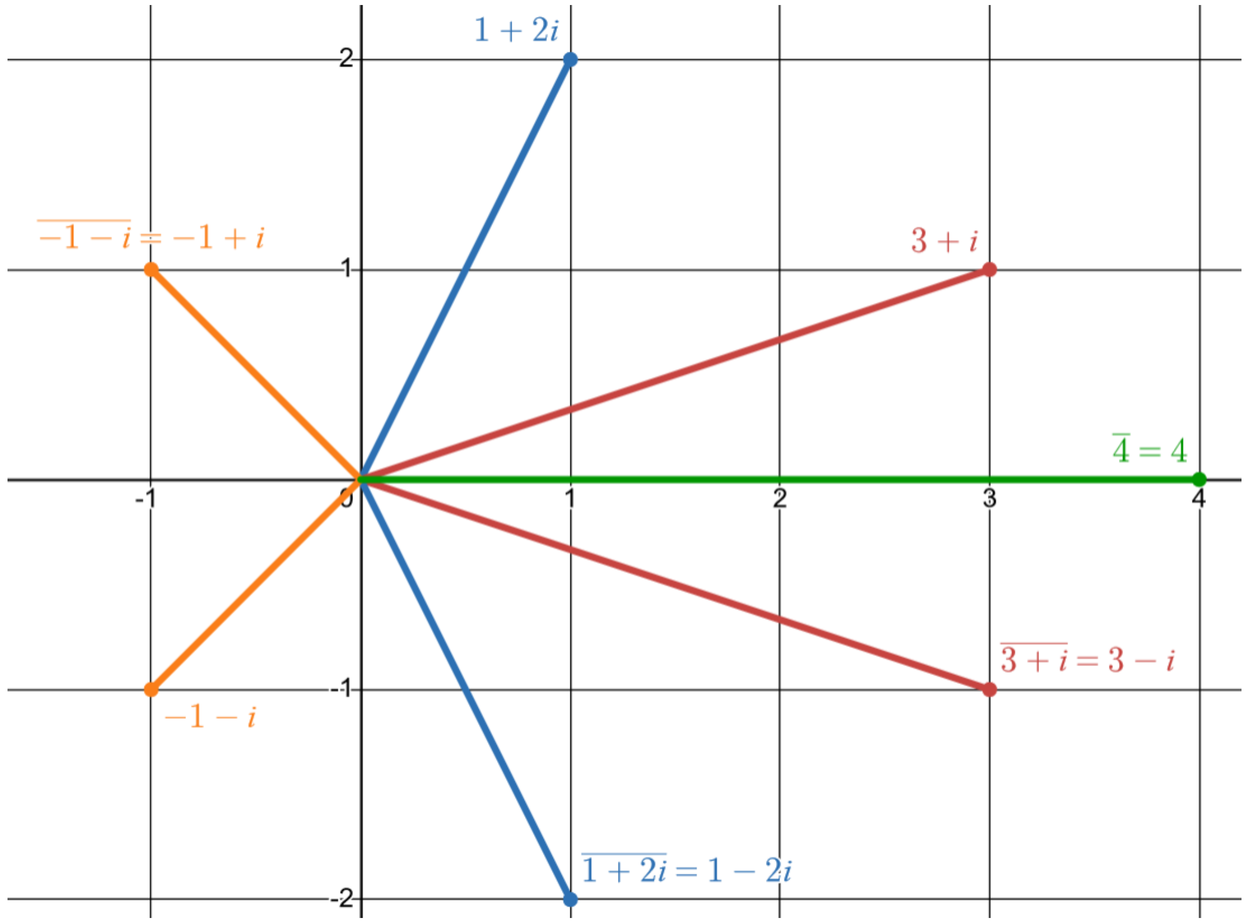
**Definition 6.** Let $z$ be a complex number expressed in Cartesian form as $a + bi$. Then the **conjugate** of $z$ is $a - bi$, and it is denoted with an overline (or bar): $\overline{z} = a - bi$.

Geometrically, $\overline{z}$ is **the reflection of** $z$ **over the real axis**. Another geometric way to view it is that the $\overline{z}$ has the same radius as $z$ but the negative angle.

That is, if $z = re^{\theta i}$, then $\overline{z} = re^{-\theta i}$.

Since reflecting over the same line twice is the same as doing nothing that means that conjugating twice is the same as doing nothing, so $\overline{\overline{z}} = z$.

Below are some illustrations of various complex numbers and their conjugates.




Namely, this shows that $\overline{3+i} = 3-i$, and $\overline{1+2i} = 1-2i$, and $\overline{-1-i} = -1+i$, and $\overline{4} = 4$. In fact, for all real numbers $a$, we have $\overline{a} = a$.

**Theorem 3.** Let $z$ be a complex number. Then

$$z + \overline{z} \in \mathbb{R}$$

and

$$z \cdot \overline{z} \in \mathbb{R}.$$

Moreover, if $z \notin \mathbb{R}$, then $\overline{z}$ is the *only* number in $\mathbb{C}$ that makes the two above statements true.

*Proof.* Let's use Cartesian coordinates: $z = a + bi$. Then

$$z + \overline{z} = (a + bi) + (a - bi) = 2a,$$

which is indeed in $\mathbb{R}$, since $a \in \mathbb{R}$. Likewise,

$$z \cdot \overline{z} = (a + bi)(a - bi) = a^2 + b^2,$$

which is again real since $a$ and $b$ are both real.

Now let $z = a + bi \in \mathbb{C}$ but $z \notin \mathbb{R}$, and suppose $c + di$ is a complex number such that $(a+bi)+(c+di)$ is real and $(a+bi)(c+di)$ is real. Then since $(a+bi)+(c+di)$ is real, that means that

$$(a+bi)+(c+di) = (a+c)+(b+d)i$$

has an imaginary component of 0, so $b + d = 0$, so $d = -b$.

On the other hand, since $(a+bi)(c+di)$ is real, we know that

$$(a+bi)(c+di) = (ac-bd)+(ad+bc)i$$

has an imaginary component of 0, so $ad + bc = 0$. But earlier we found $d = -b$, so subbing that in gives $-ab + bc = 0$, so $b(-a + c) = 0$. Now importantly: since $z \notin \mathbb{R}$, that means that the imaginary component of $z$ is not zero, namely $b \neq 0$. So we can divide both sides of the equation by $b$ and get $-a + c = 0$, so $a = c$. So since $a = c$ and $d = -b$, that means $c + di = a - bi$, which is the conjugate of $z$. □

The conjugate is useful because then we can divide complex numbers without having to convert to polar; we can instead multiply and divide by the conjugate of the denominator. So we can write

$$\frac{1}{a+bi} = \frac{1}{a+bi} \cdot \frac{a-bi}{a-bi} = \frac{a-bi}{a^2+b^2} = \frac{a}{a^2+b^2} - \frac{bi}{a^2+b^2}.$$

For example:

$$\frac{1}{1+i} = \frac{1-i}{1^2+1^2} = \frac{1}{2} - \frac{1}{2}i.$$

Another example,

$$\frac{2+3i}{4-5i} = \frac{(2+3i)(4+5i)}{(4-5i)(4+5i)} = \frac{-7+22i}{41} = -\frac{7}{41} + \frac{22}{41}i.$$

Conjugates also give us the following important theorem.

**Theorem 4.** Let $f(x)$ be a polynomial with real coefficients, and let $z$ be a root of $f(x)$. Then $\overline{z}$ is a root of $f(x)$.

In order to prove this theorem, we will need the help of the following lemma:

**Lemma 1.** Let $z_1, z_2 \in \mathbb{C}$. Then

$$\overline{z_1 + z_2} = \overline{z}_1 + \overline{z}_2$$

and

$$\overline{z_1 z_2} = \overline{z}_1 \overline{z}_2$$

and

$$\overline{z_1^n} = (\overline{z}_1)^n$$

for all powers $n$.

*Proof of Lemma 1.* Let $z_1 = a + bi$ and $z_2 = c + di$. Then

$$\overline{z_1 + z_2} = \overline{(a + c) + (b + d)i} = (a + c) - (b + d)i = (a - bi) + (c - di) = \overline{z}_1 + \overline{z}_2.$$

Next,

$$\overline{z_1 z_2} = \overline{(ac - bd) + (ad + bc)i} = (ac - bd) - (ad + bc)i,$$

and

$$\overline{z}_1 \overline{z}_2 = (a - bi)(c - di) = (ac - bd) + (-ad - bc)i = (ac - bd) - (ad + bc)i,$$

which is the same thing we got from $\overline{z_1 z_2}$, so $\overline{z_1 z_2} = \overline{z}_1 \overline{z}_2$.

Note that as a result of the fact that $\overline{z_1 z_2} = \overline{z}_1 \overline{z}_2$, we have

$$\overline{z_1^2} = \overline{z}_1 \overline{z}_1 = (\overline{z}_1)^2.$$

Then we can use a technique called **proof by induction** to prove this for even higher powers: suppose

$$\overline{z_1^n} = (\overline{z}_1)^n$$

is true for some $n$ (like it is for $n = 2$). Then let's take the $n + 1$ power.

$$\overline{z_1^{n+1}} = \overline{z_1 z_1^n} = \overline{z}_1 \overline{z_1^n} = \overline{z}_1 (\overline{z}_1)^n = (\overline{z}_1)^{n+1},$$

making the lemma true for $n + 1$ as well. (I go into more detail on proofs by induction in the class notes for Midterm 2.) □

Now we are ready to prove the theorem.

*Proof of Theorem 4.* Let

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

be a polynomial and $a_j \in \mathbb{R}$ for all $j$. Then suppose $z \in \mathbb{C}$ and $f(z) = 0$, so $z$ is a root of $f(x)$. Then

$$a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 = 0.$$

Then let's see what happens when we plug $\overline{z}$ into $f(x)$.

$$f(\overline{z}) = a_n \overline{z}^n + a_{n-1} \overline{z}^{n-1} + \cdots + a_1 \overline{z} + a_0.$$

Using Lemma 1, $\overline{z}^j = \overline{z^j}$ for all $j$, so

$$f(\overline{z}) = a_n \overline{z^n} + a_{n-1} \overline{z^{n-1}} + \cdots + a_1 \overline{z} + a_0.$$

Next, since $a_j \in \mathbb{R}$ for all $j$, that means $a_j = \overline{a}_j$, so we can write

$$f(\overline{z}) = \overline{a}_n \overline{z^n} + \overline{a}_{n-1} \overline{z^{n-1}} + \cdots + \overline{a}_1 \overline{z} + \overline{a}_0.$$

Next, we can use Lemma 1 again to rewrite $(\overline{a}_j)(\overline{z^j})$ as $\overline{a_j z^j}$.

$$f(\overline{z}) = \overline{a_n z^n} + \overline{a_{n-1} z^{n-1}} + \cdots + \overline{a_1 z} + \overline{a}_0.$$

Finally, we can use Lemma 1 one last time to rewrite this all as

$$f(\overline{z}) = \overline{a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0} = \overline{f(z)} = \overline{0} = 0,$$

so we finally have

$$f(\overline{z}) = 0$$

which means $\overline{z}$ is a root of $f(x)$. □

# 6 Back to Spectral Theory

Now let's take another look at the matrix from Section 1:

$$A = \begin{bmatrix} 1 & 1 \\ -3 & -2 \end{bmatrix}.$$

Taking a look at the initial state vector $\vec{v}_0 = \begin{bmatrix} 400 \\ 200 \end{bmatrix}$, let's compute some of the subsequent states.

$$\vec{v}_1 = \begin{bmatrix} 1 & 1 \\ -3 & -2 \end{bmatrix} \begin{bmatrix} 400 \\ 200 \end{bmatrix} = \begin{bmatrix} 400 + 200 \\ -1200 - 400 \end{bmatrix} = \begin{bmatrix} 600 \\ -1600 \end{bmatrix}.$$

$$\vec{v}_2 = \begin{bmatrix} 1 & 1 \\ -3 & -2 \end{bmatrix} \begin{bmatrix} 600 \\ -1600 \end{bmatrix} = \begin{bmatrix} 600 - 1600 \\ -1800 + 3200 \end{bmatrix} = \begin{bmatrix} -1000 \\ 1400 \end{bmatrix}.$$

$$\vec{v}_3 = \begin{bmatrix} 1 & 1 \\ -3 & -2 \end{bmatrix} \begin{bmatrix} -1000 \\ 1400 \end{bmatrix} = \begin{bmatrix} -1000 + 1400 \\ 3000 - 2800 \end{bmatrix} = \begin{bmatrix} 400 \\ 200 \end{bmatrix} = \vec{v}_0,$$

and so we are back where we started after three moves! Let's use some spectral theory to take a look under the hood of this matrix.

Computing the characteristic polynomial for $A$, we get

$$c_A(\lambda) = \det \begin{bmatrix} 1-\lambda & 1 \\ -3 & -2-\lambda \end{bmatrix} = (1-\lambda)(-2-\lambda) + 3 = \lambda^2 + \lambda + 1.$$

Using the quadratic formula, we can find the roots.

$$\lambda = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm i\sqrt{3}}{2}.$$

(Note that this matches what Theorem 4 says: the two roots are conjugates of each other!)

Notice that these numbers are a little familiar. Let's convert them into polar coordinates: computing the radii we get

$$r = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{\pm\sqrt{3}}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1.$$

And computing the angles gives us

$$\theta = \pi + \arctan\left(\pm\sqrt{3}\right) = \begin{cases} \pi + \arctan(\sqrt{3}) = 4\pi/3 \\ \pi + \arctan(-\sqrt{3}) = 2\pi/3 \end{cases}.$$

So the polar forms of the eigenvalues are $e^{\frac{2\pi i}{3}}$ and $e^{\frac{4\pi i}{3}}$. These are two of the third roots of unity, $\omega_3$ and $\omega_3^2$! That means

$$A \sim \underbrace{\begin{bmatrix} \omega_3 & 0 \\ 0 & \omega_3^2 \end{bmatrix}}_{D}.$$

So now we know that $\lim_{t\to\infty} A^t$ does not exist. This is because $\lim_{t\to\infty} D^t$ does not exist, because $\lim_{t\to\infty} \omega_3^t$ and $\lim_{t\to\infty} \omega_3^{2t}$ don't exist. But these sequences can be broken into three branches! That is $\lim_{t\to\infty} A^{3t}$, $\lim_{t\to\infty} A^{3t+1}$, and $\lim_{t\to\infty} A^{3t+2}$ all exist. This is because

$$\lim_{t\to\infty} \omega_3^{3t} = \lim_{t\to\infty} 1^t = 1,$$

$$\lim_{t\to\infty} \omega_3^{3t+1} = \lim_{t\to\infty} 1^t \omega_3 = \omega_3,$$

and

$$\lim_{t\to\infty} \omega_3^{3t+2} = \lim_{t\to\infty} 1^t \omega_3^2 = \omega_3^2.$$

For $\omega_3^2$, we have

$$\lim_{t\to\infty} \left(\omega_3^2\right)^{3t} = \lim_{t\to\infty} \omega_3^{6t} = \lim_{t\to\infty} 1^t = 1,$$

$$\lim_{t\to\infty} \left(\omega_3^2\right)^{3t+1} = \lim_{t\to\infty} 1^t \omega_3^2 = \omega_3^2,$$

and

$$\lim_{t\to\infty} \left(\omega_3^2\right)^{3t+2} = \lim_{t\to\infty} 1^t \omega_3^4 = \omega_3^4 = \omega_3.$$