New Perspectives on Geproci-ness

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15 April 2023

What is Geproci?

Definition

A finite set Z in $\mathbb{P}^n_k$ is **geproci** if the projection $\overline{Z}$ of Z from a general point P to a hyperplane is a complete intersection in $\mathbb{P}^{n-1}_k$.

Geproci stands for **ge**neral **pro**jection is a **c**omplete **i**ntersection.

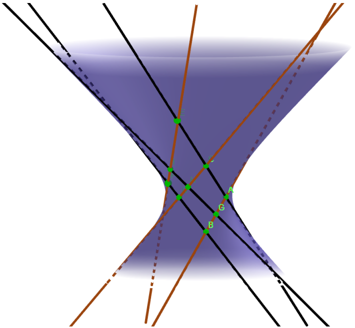
The only nontrivial examples known are for $n = 3$. In this case a hyperplane is a plane H. A reduced set of points in a plane is a complete intersection if it is the transverse intersection of two algebraic curves, like this.

For $\#Z = ab$ $(a \leq b)$, Z is (a, b)-geproci if $\overline{Z}$ is the intersection of a degree a curve and a degree b curve.

Trivial Cases: Coplanar Points and Grids

A set of coplanar points can be geproci if and only if they are already a complete intersection in the plane they're on.

The easiest non-coplanar examples are grids.

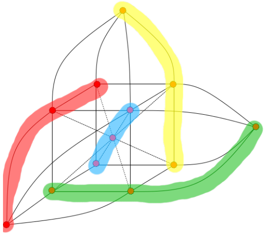


Summary of Nontrivial Cases

Half-Grids: A procedure is known for creating an (a, b)-geproci half-grid for $4 \leq a \leq b$, but it is not known what other examples there can be.

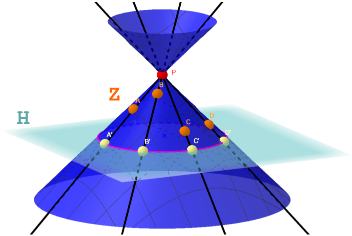
Non-Half-Grids: Before my thesis work, only a few examples were known and there was no known way to generate more.

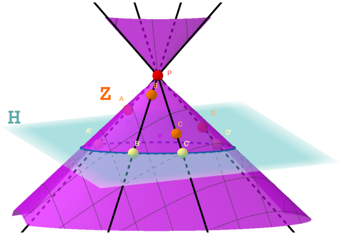
Because of this, non-half-grids have been mysterious; it's easier to get an idea of what a half-grid is like.



Cones and Geproci

It is of interest when there is a cone through Z whose vertex is a general point P, and which meets H in a curve containing the projected image of Z. For Z to be (a, b)-geproci, there needs to be two such cones, of degrees a, b.





Geometry in Positive Characteristic

Geometry gets weird in positive characteristic p! For example, there's Fermat's Little Theorem and there's the Freshman's Dream (aka Frobenius): $(x+y)^p = x^p + y^p$. But this weirdness makes being geproci very natural!

Cones in $\mathbb{P}^3_{\mathbb{F}_q}$ of degree $a = q + 1$

Consider $Z = \mathbb{P}^3_{\mathbb{F}_q}$.

Note that $\#Z = \dfrac{q^4 - 1}{q - 1} = q^3 + q^2 + q + 1 = (q + 1)(q^2 + 1)$.

There is a degree $q + 1$ cone containing Z whose vertex is at a general point $P = (a, b, c, d) \in \mathbb{P}^3_k$, $k = \overline{\mathbb{F}}_q$. This cone is given by

$$\begin{vmatrix} a & b & c & d \\ a^q & b^q & c^q & d^q \\ x & y & z & w \\ x^q & y^q & z^q & w^q \end{vmatrix} = 0$$

Spreads in $\mathbb{P}^3_{\mathbb{F}_q}$

Is there a cone of degree $b = q^2 + 1$? There is!

Each line of $\mathbb{P}^3_{\mathbb{F}_q}$ contains $q + 1$ points. Can $\mathbb{P}^3_{\mathbb{F}_q}$ be partitioned by $q^2 + 1$ mutually-skew lines? Yes! Such a partition is called a **spread**, a name from combinatorics. Over $\mathbb{R}$, the Hopf fibration gives an example of a spread in $\mathbb{P}^3_{\mathbb{R}}$.

$$\begin{array}{ccc} S^3 & \xrightarrow{H} & S^2 \\ \downarrow A & & \downarrow = \\ \mathbb{P}^3_{\mathbb{R}} & \xrightarrow{F} & \mathbb{P}^1_{\mathbb{C}} \\ \cup & & \Cup \\ \mathbb{P}^1_{\mathbb{R}} & \xrightarrow{F} & * \end{array}$$

The join of each line of the spread with P is our cone.

A Theorem

The following result gives a new method of constructing nontrivial geproci sets.

Theorem (K–)

The set of points $\mathbb{P}^3_{\mathbb{F}_q}$ *is* $(q+1, q^2+1)$*-geproci in* $\mathbb{P}^3_k$*, where* k *is an algebraically closed field containing* $\mathbb{F}_q$*.*

Note when $q = 2$, we get a non-trivial $(3, 5)$-geproci set! These do not exist in characteristic 0 [CFFHMSS].

Partial Spreads

Definition

A **partial spread** of $\mathbb{P}^3_{\mathbb{F}_q}$ with deficiency d is a set of $q^2 + 1 - d$ mutually-skew lines. A **maximal partial spread** is a partial spread of positive deficiency that is not contained in any larger partial spread.

Complements of Maximal Partial Spreads

Maximal partial spreads give a way of producing infinitely many nontrivial non-half-grids.

Theorem (K–)

The complement of a maximal partial spread of deficiency d is a non-trivial $\{q+1, d\}$-geproci set. Furthermore, when $d > q+1$, the complement is a non-trivial non-half-grid.

In 1993 and 2002, Heden proved that there are maximal partial spreads of every deficiency in the interval $q+1 < d < \frac{q^2+1}{2} - 6$.

Bibliography

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