New perspectives on geproci sets

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What is Geproci?

Definition

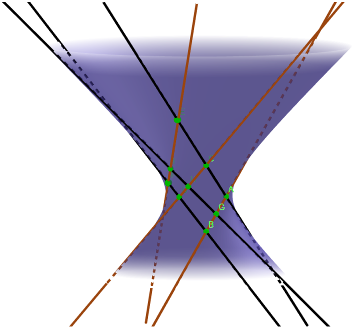
A finite set Z in $\mathbb{P}^n_k$ is **geproci** if the projection $\overline{Z}$ of Z from a general point P to a hyperplane $H = \mathbb{P}^{n-1}_k$ is a complete intersection in H.

Geproci stands for **ge**neral **pro**jection is a **c**omplete **i**ntersection.
The only nontrivial examples known are for $n = 3$. In this case a hyperplane is a plane. A reduced set of points in a plane is a complete intersection if it is the transverse intersection of two algebraic curves.
For $\#Z = ab$ ($a \leq b$), Z is (a, b)-geproci if $\overline{Z}$ is the intersection of a degree a curve and a degree b curve.

Trivial Cases: Coplanar Points and Grids

A set of coplanar points is geproci if and only if it is a complete intersection on the plane it's in.

The easiest non-coplanar examples are grids, which are sets of points that form the intersection of two families of mutually-skew lines.

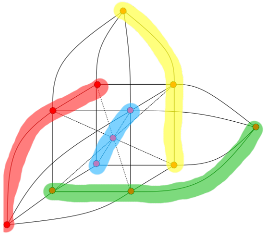


Summary of Nontrivial Cases

Half-Grids: A procedure is known for creating an (a, b)-geproci half-grid for $4 \leq a \leq b$, but it is not known what other examples there can be.

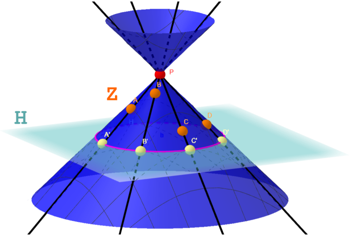
Non-Half-Grids: Before my thesis work, only a few examples were known and there was no known way to generate more.

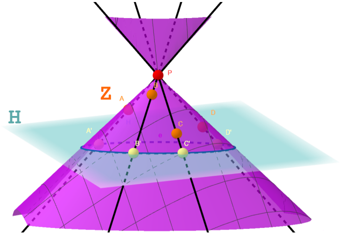
Because of this, nontrivial non-half-grids have been mysterious; it's easier to get an idea of what a half-grid is like.



Cones and Geproci

It is of interest when there is a cone through Z whose vertex is a general point P, and which meets H in a curve containing the projected image of Z. For Z to be (a, b)-geproci, there needs to be two such cones, of degrees a, b.





Geometry gets weird in positive characteristic p! For example, there's Fermat's Little Theorem and there's the Freshman's Dream (aka Frobenius): $(x+y)^p = x^p + y^p$. But this weirdness makes being geproci very natural!

Consider $Z = \mathbb{P}^3_{\mathbb{F}_q}$.

Note that $\#Z = \dfrac{q^4 - 1}{q - 1} = q^3 + q^2 + q + 1 = (q + 1)(q^2 + 1)$.

There is a unique degree $q + 1$ cone containing Z whose vertex is at a general point $P = (a, b, c, d) \in \mathbb{P}^3_k$, $k = \overline{\mathbb{F}}_q$. This cone meets every line through two points of $\mathbb{P}^3_{\mathbb{F}_q}$ transversely. It is given by

$$\begin{vmatrix} a & b & c & d \\ a^q & b^q & c^q & d^q \\ x & y & z & w \\ x^q & y^q & z^q & w^q \end{vmatrix} = 0$$

Is there a cone of degree $b = q^2 + 1$? There is!
Each line of $\mathbb{P}^3_{\mathbb{F}_q}$ contains $q + 1$ points. Can $\mathbb{P}^3_{\mathbb{F}_q}$ be partitioned by mutually-skew lines? Yes! Such a partition is called a **spread**, a name from combinatorics. The fibers S^1 of the Hopf fibration H map to the fibers $\mathbb{P}^1_{\mathbb{R}}$ of F, which give an example of a spread in $\mathbb{P}^3_{\mathbb{R}}$.

$$\begin{array}{ccc} S^3 & \xrightarrow{H} & S^2 \\ \downarrow A & & \downarrow = \\ \mathbb{P}^3_{\mathbb{R}} & \xrightarrow{F} & \mathbb{P}^1_{\mathbb{C}} \end{array}$$

For $\mathbb{P}^3_{\mathbb{F}_q}$, there are $q^2 + 1$ lines in the spread. The join of each line of the spread with P is our cone.

A Theorem

The following result gives a new method of constructing nontrivial geproci sets.

Theorem (K–)

The set of points $\mathbb{P}^3_{\mathbb{F}_q}$ is $(q+1, q^2+1)$-geproci in $\mathbb{P}^3_k$, where k is an algebraically closed field containing $\mathbb{F}_q$.

Note when $q = 2$, we get a non-trivial $(3, 5)$-geproci set! No nontrivial $(3, 5)$-geproci set exists in characteristic 0 [CFFHMSS], so this is new.

Partial Spreads

Definition

A **partial spread** of $\mathbb{P}^3_{\mathbb{F}_q}$ with **deficiency** d is a set of $q^2 + 1 - d$ mutually-skew lines. A **maximal partial spread** is a partial spread of positive deficiency that is not contained in any larger partial spread.

Complements of Maximal Partial Spreads

Maximal partial spreads give a way of producing infinitely many nontrivial non-half-grids.

Theorem (K–)

The complement of a maximal partial spread of deficiency d is a non-trivial $\{q+1, d\}$-geproci set. Furthermore, when $d > q+1$, the complement is a non-trivial non-half-grid.

In 1993 and 2002, Heden proved for $q \geq 7$ that there are maximal partial spreads of every deficiency d in the interval $q-1 \leq d \leq \frac{q^2+1}{2} - 6$.

The field $\mathbb{F}_7$ and Gorenstein Configurations

The fifteen maximal partial spreads of size 45 in $\mathbb{P}^3_{\mathbb{F}_7}$ were classified by Soicher in 2000. Their complements are configurations of 40 points. Each complement is $(5, 8)$-geproci and is a non-half-grid. Furthermore, at least four of the fifteen are different up to projective equivalence and are Gorenstein! The four configurations that have been tested so far have stabilizers in $PGL(4, 7)$ of different sizes (10, 20, 60, and 120) and so are not projectively equivalent.

In characteristic 0, only one non-trivial Gorenstein configuration is known up to projective equivalence, also a configuration of 40 points (the Penrose configuration). [CFFHMSS].

Infinitely-Near Points

Definition

Let X be a smooth algebraic variety and let $P \in X$. The point Q is **infinitely-near** P if Q is on the exceptional locus of the blowup of X at P. (Intuitively, Q is a tangent direction at P.)

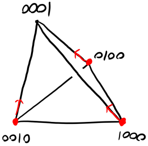
Abuse of notation: Technically, $Q \in \mathrm{BL}_P(X)$, but it is traditional to speak of infinitely-near points as if they were points of X itself.

Geproci With Infinitely-Near Points

Theorem (K–)

Let char $k = 2$. Let $Z = \{(1,0,0,0) \times 2, (0,1,0,0) \times 2, (0,0,1,0) \times 2\}$ (where $p_i \times 2$ represents an ordinary point $p_i \in \mathbb{P}^3_k$ and a point q_i infinitely near p_i), with the infinitely-near point at each ordinary point corresponding to the tangent along the line through p_i and $(0,0,0,1)$. Then Z is a $(2,3)$-geproci half-grid.

No $(2,3)$ half-grid is known in characteristic 0.

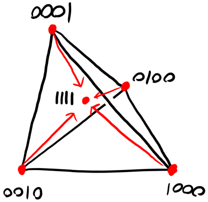


Another Example

Theorem (K–)

Let
$Z = \{(1,0,0,0)\times 2, (0,1,0,0)\times 2, (0,0,1,0)\times 2, (0,0,0,1)\times 2, (1,1,1,1)\}$, *with each infinitely-near point corresponding to the line containing* $(1,1,1,1)$. *Then* Z *is* $(3,3)$*-geproci. It is a non-trivial non-half-grid.*

No nontrivial $(3,3)$-geproci sets are known in characteristic 0.



Bibliography

L. Chiantini, L. Farnik, G. Favacchio, B. Harbourne, J. Migliore, T. Szemberg, and J. Szpond. Configurations of points in projective space and their projections. arXiv:2209.04820, 2022.

O. Heden. Maximal partial spreads and the modular n-queen problem III. Discrete Mathematics, 243:135–150, 2002.

O. Heden. Maximal partial spreads and the modular n-queen problem. Discrete Mathematics, 120:75–91, 1993.