# TOPICS

1. **Definitions**: *order, linear, autonomous, normal form, separable, standard form, linear dependence/independence, homogeneous, differential operators/annihilators*

   (a) Order: The *order* of a differential equation is the highest-taken derivative in the equation.

   i. The order of $6y^{(5)} - \sin(x)\left(y^{(4)}\right)^2 + 3e^y y''' = \arctan(x)$ is 5.

   ii. The order of $2^x \frac{d^3y}{dx^3} + \sin(x)\frac{d^2y}{dx^2} = y - \frac{d^4y}{dx^4}$ is 4.

   (b) Linear: a differential equation is *linear* if each derivative of the dependent variable is in its own term, is not raised to a power other than 1, and is not inside any function. That is, a DE is linear if it can be written in the form

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x),$$

   where the functions $a_i(x)$ and $g(x)$ are functions involving only the independent variable (in this case, $x$).

   i. Linear: $\sin^2(x)\frac{d^5y}{dx^5} + e^{5x^2-6x+10}\frac{d^4y}{dx^4} + \frac{d^2y}{dx^2} - 15\frac{dy}{dx} + \arctan(x-1)y = \cos(x)$.

   ii. Linear: $11y + y''' + e^x y' - 18xy'' + \sin(x)y' - 12x = 9x$. Note: it doesn't matter that the derivatives are not written in highest-to-lowest order, or that $-12x$ appears on one side and $9x$ appears on the other. This DE can be re-written as $y''' - 18xy'' + \sin(x)y' + 11y = 21x$ if we want to put it in a nicer form that is more clearly linear.

   iii. Linear: $\sin(x)y'' + 18xy'' - \ln(x)y' + 9y + 11x = \arctan(x)$. Note: we can re-write this differential equation as $(\sin(x) + 18x)y'' - \ln(x)y' + 9y = \arctan(x) - 11x$ in order to put it in a nicer form.

   iv. NOT Linear: $y^{(4)} + y''' + y'' + y' + \sin(y) = 0$. The $\sin(y)$ is an issue.

   v. NOT Linear: $y(y'' + 3y') + 3(y''')^2 = \cos(x)$. Both the $y(y'' + 3y')$ and the $(y''')^2$ prevent it from being linear.

   (c) Autonomous: a <u>first-order</u> DE is *autonomous* if it does not involve the dependent variable at all.

   i. Autonomous: $\frac{dy}{dx} = e^y$.

   ii. Autonomous: $y' - \sin(y) = 10 - \cos(y)$.

   iii. Not Autonomous: $y' = y + x$.

   iv. Not Autonomous: $xy' - 15xy = \sin(x)$.

   (d) Normal form: a differential equation is written in *normal form* if the highest-order derivative is on its own side of the equal sign.

i. The DE $y''' + 15xy'' - \sin(x)y' + 11xy = 100x$ can be re-written into normal form as

$$y''' = -15xy + \sin(x)y' - 11xy + 100x.$$

ii. The DE $4xy'''' - \arcsin(x)y''' + 10y'' - 17y = e^{x^2+1}$ can be written into normal form as

$$y'''' = \frac{\arcsin(x)y''' - 10y'' + 17y + e^{x^2+1}}{4x}.$$

(e) Separable: a first-order DE is separable if it can be written in the form $y' = h(y)g(x)$, where $h$ is a function just involving $y$ and $g$ is a function just involving $x$.

i. Separable: $y' = \sin(x)\cos(y)$. Here, $h(y) = \cos(y)$ and $g(x) = \sin(x)$.

ii. Separable: $2xy' - x^2y = 0$. This can be re-written as $y' = \frac{1}{2}xy$, and we can put $h(y) = y$ and $g(x) = \frac{1}{2}x$.

iii. Not Separable: $y' = y\sin(x) + \cos(y)$

iv. Not Separable: $xy' + x^2y - 10x = 0$.

Separable equations can be solved using the **separation of variables** method, detailed later.

(f) Standard form: a first-order linear DE is in *standard form* if it is in the form $y' + P(x)y = f(x)$, where $P(x)$ and $f(x)$ are functions involving only the independent variable (in this case, $x$).

i. The standard form of $xy' - 15xy = \sin(x)$ is $y' - 15y = \frac{\sin(x)}{x}$.

ii. The standard form of $y' = -\cos(x)y + 11e^x$ is $y' + \cos(x)y = 11e^x$.

Writing a first-order linear DE in standard form is the first step in using the **integrating factors** method, detailed later.

(g) Linear Dependence/Independence: a set of functions $y_1, \ldots, y_k$ is *linearly dependent* if there are some constants $c_1, \ldots, c_k$ **with at least one not equalling zero** such that

$$c_1y_1 + c_2y_2 + \cdots + c_ky_k = 0.$$

The set $y_1, \ldots, y_k$ are *linearly independent* if the ONLY way the linear combination $c_1y_1 + c_2y_2 + \cdots + c_ky_k$ can be zero is if ALL of the coefficients $c_i$ are zero.

i. $x^a$ and $x^b$ are linearly independent as long as $a \neq b$.

ii. $e^{ax}$ and $e^{bx}$ are linearly independent as long as $a \neq b$.

iii. The three functions $x$, $x^2$, and $x^2 + x$ are linearly dependent, because for example

$$1 \cdot x + 1 \cdot x^2 + (-1) \cdot (x^2 + x) = 0,$$

but the coefficients are 1, 1 and $-1$, which are not zero. If you removed any one of the functions from the set, then the remaining two would be linearly independent: $x$ and $x^2$ are linearly independent, $x$ and $x^2 + x$ are linearly independent, and $x^2$ and $x^2 + x$ are linearly independent.

iv. Any set of functions that includes 0 is automatically linearly dependent.

v. $\cos(ax)$ and $\cos(bx)$ are linearly independent as long as $a \neq b$.

vi. $\sin(ax)$ and $\sin(bx)$ are linearly independent as long as $a \neq b$, $a \neq 0$, and $b \neq 0$.

vii. $\cos(ax)$ and $\sin(bx)$ are linearly independent as long as $b \neq 0$.

(h) Homogeneous: a <u>linear</u> DE is *homogenous* if it can be written in the form

$$a_n(x)\frac{\mathrm{d}^n y}{\mathrm{d}x^n} + a_{n-1}(x)\frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}} + \cdots + a_2(x)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0(x)y = 0.$$

Homogeneous differential equations follow the *superposition principle*: if $y_1, \ldots, y_k$ are solutions, then any *linear combination* of $y_1, \ldots, y_k$ are also solutions. A linear combination is a function of the form

$$y = c_1 y_1 + c_2 y_2 + \cdots + c_k y_k,$$

where $c_1, \ldots, c_k$ are constants.

A set of $n$ linearly independent solutions $y_1, \ldots, y_n$ to an $n^{\text{th}}$-order linear homogeneous differential equation form a **fundamental solution set**: the general solution to the equation takes the form $y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$.

When an $n^{\text{th}}$-order linear differential equation is NOT homogeneous, the general solution is $y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n + y_p$, where $y_1, \ldots, y_n$ form the fundamental solution set of the corresponding homogeneous DE, and $y_p$ is a particular solution of the DE.

(i) Differential Operators/Annihilators: the *differential operator* $D$ is a function that receives the input of a function $f(x)$ and outputs its derivative $f'(x)$. We can take powers of $D$ to create higher-order differential operators: $D^k(f(x)) = \dfrac{d^k f}{dx^k}$ for all $k \geq 0$. Note $D^0(f(x)) = f(x)$ (the 0th derivative). We can also form linear combinations of powers of differential operators, for example $(5D^2 + 6D + 11D^0)(y) = 5y'' + 6y' + 11y$.

A linear combination of differential operators $L$ *annihilates* a function $f(x)$ if $L(f(x)) = 0$.

i. The lowest-degree annihilator of $x^n$ is $D^{n+1}$: $D^{n+1}$ receives a function $f(x)$ and outputs the $n+1$ derivative, so $D^{n+1}(x^n) = 0$.

ii. $D-1$ annihilates $e^x$: $(D-1)(e^x) = D(e^x) - 1(e^x) = e^x - e^x = 0$.

iii. In general, $e^{rx}$ is annihilated by $(D-r)$ and $x^n e^{rx}$ is annihilated by $(D-r)^{n+1}$.

iv. $\cos(bx)$ and $\sin(bx)$ are both annihilated by $D^2+b^2$: $(D^2+b^2)(\cos(bx)) = D^2(\cos(bx)) + b^2\cos(bx) = -b^2\cos(bx) + b^2\cos(bx) = 0$. The same goes for $\sin(bx)$. In general, $x^n\cos(bx)$ and $x^n\sin(bx)$ are annihilated by $(D^2+b^2)^{n+1}$.

v. In general, $e^{ax}\cos(bx)$ and $e^{ax}\sin(bx)$ are annihilated by $D^2 - 2aD + (a^2+b^2)$. Also $x^n e^{ax}\cos(bx)$ are $x^n e^{ax}\sin(bx)$ are annihilated by $(D^2 - 2aD + (a^2+b^2))^{n+1}$.

2. **Existence and Uniqueness Theorems**

There are two existence/uniqueness theorems that we've seen: one for *any kind* of first-order initial-value problem (IVP), and one for *linear* IVPs of any order.

(a) The first existence/uniqueness theorem regards first-order IVPs of the form

$$\begin{aligned} \text{DE :} & \frac{dy}{dx} = f(x,y) \\ \text{IC :} & y(x_0) = y_0. \end{aligned}$$

There is a unique solution to this IVP on an interval $I = (x_0 - h, x_0 + h)$ if there is a rectangular (possibly infinite) region $R$ in the $xy$-plane containing $I$ such that $f(x,y)$ and $\dfrac{\partial f}{\partial y}$ are continuous on $R$.

(b) The second existence/uniqueness theorem regards linear IVPs of the form

$$\begin{aligned} \text{DE :} & a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \\ \text{ICs :} & y(x_0) = y_1 \\ & y'(x_0) = y_1 \\ & y''(x_0) = y_2 \\ & \vdots \\ & y^{(n-1)}(x_0) = y_{n-1}. \end{aligned}$$

There is a unique solution to this IVP on an interval $I$ containing $x_0$ if all of the $a_i(x)$'s and $g(x)$ are continuous on $I$ and if $a_n(x) \neq 0$ for any $x$ in $I$.

3. **Slope Fields**

4. **Modeling Real-World Phenomena**

   (a) Salt-Water Solutions

   (b) Logistic Growth

   (c) Newton's Law of Heating and Cooling

5. **Euler's Method**

   If we are given a first-order IVP with the DE $y' = f(x, y)$ and the IC $y(x_0) = y_0$, we can construct a sequence of points that approximate a solution by choosing an increment $h$ and constructing

$$
\begin{aligned}
&(x_0, y_0) \\
&(x_1, y_1) = (x_0 + h, y_0 + h \cdot f(x_0, y_0)) \\
&(x_2, y_2) = (x_1 + h, y_1 + h \cdot f(x_1, y_1)) \\
&(x_3, y_3) = (x_2 + h, y_2 + h \cdot f(x_2, y_2)) \\
&\quad\vdots \\
&(x_n, y_n) = (x_{n-1} + h, y_{n-1} + h \cdot f(x_{n-1}, y_{n-1})) \\
&\quad\vdots
\end{aligned}
$$

   Choosing a smaller value for $h$ results in better approximations.

6. **Separation of Variables**

   This is a method of solving first-order separable differential equations. Given a differential equation

$$\frac{\mathsf{d}y}{\mathsf{d}x} = h(y)g(x),$$

   we can re-write this as

$$\frac{\mathsf{d}y}{h(y)} = g(x)\mathsf{d}x$$

and integrate both sides to get a solution

$$\int \frac{\mathsf{d}y}{h(y)} = G(x) + C,$$

where $G(x)$ is an antiderivative of $g(x)$ and $C$ is a constant.

7. **Integrating Factors**

This is a method of solving first-order linear differential equations.

(a) The first step is to get such an equation into standard form:

$$\frac{\mathsf{d}y}{\mathsf{d}x} + P(x)y = f(x).$$

(b) Next we want to find the integrating factor:

$$\mu(x) = e^{\int P(x)\mathsf{d}x}.$$

(c) Next we want to multiply both sides of the equation (still using standard form) by $\mu(x)$:

$$\mu(x)\left(\frac{\mathsf{d}y}{\mathsf{d}x} + P(x)y\right) = \mu(x)f(x).$$

The left side can be re-written as

$$\frac{\mathsf{d}}{\mathsf{d}x}(\mu(x)y).$$

So our equation turns into

$$\frac{\mathsf{d}}{\mathsf{d}x}(\mu(x)y) = \mu(x)f(x).$$

(d) The last step is to integrate both sides.

$$\int \frac{\mathsf{d}}{\mathsf{d}x}(\mu(x)y)\mathsf{d}x = \int \mu(x)f(x)\mathsf{d}x$$

and so

$$\mu(x)y = \int \mu(x)f(x)\mathsf{d}x$$

and we get

$$y = \frac{\int \mu(x)f(x)\mathsf{d}x}{\mu(x)}.$$

8. **Reduction of Order**

   If we have a second-order linear homogeneous differential equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$$

   and we are given one of the solutions $y_1$, we can find an linearly independent solution $y_2$ by setting $y_2 = u(x) \cdot y_1(x)$, plugging $uy_1$ in for $y$:

$$a_2(x)(uy_1)'' + a_1(x)(uy_1)' + a_0(x)(uy_1) = 0$$

   and solving for $u(x)$, which will give us $y_2$.

9. **Auxiliary Equations** Given a linear homogeneous DE with constant coefficients

$$a_n y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_2 y'' + a_1 y' + a_0 y = 0,$$

   (where the $a_i$'s are constants instead of functions) we can associate it to a polynomial *auxiliary equation*

$$a_n m^n + a_{n-1}m^{n-1} + \cdots a_2 m^2 + a_1 m + a_0 = 0.$$

   The roots of the auxiliary equation correspond to solutions of the differential equation in the following way:

   (a) If $r$ is a real root of multiplicity 1, we get the solution $e^{rx}$

   (b) If $r$ is a real root of multiplicity $p$, we get the solutions $e^{rx}, xe^{rx}, x^2e^{rx}, \ldots, x^{p-1}e^{rx}$.

   (c) If $a+bi$ and $a-bi$ are non-real roots of multiplicity 1, then we get the solutions $e^{ax}\cos(bx)$ and $e^{ax}\sin(bx)$.

   (d) If $a+bi$ and $a-bi$ are non-real roots of multiplicity $p$, then we get the solutions $e^{ax}\cos(bx), xe^{ax}\cos(bx), \ldots, x^{p-1}e^{ax}\cos(bx)$ and $e^{ax}\sin(bx), xe^{ax}\sin(bx), \ldots, x^{p-1}e^{ax}\sin(bx)$.